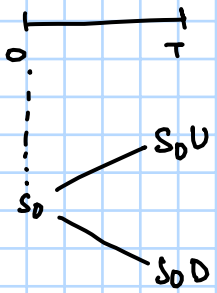
4th march:

Binomial model:

It is a discrete time model

Single step metod:



$t_0 = 0$: present time
$t_1 > t_0 = 0$: future time

given the unit market price S_0 of stock S_1 to study

1-step binomial: (u_0, d_0) pair of multiplicative factors s.t

$$0 < d_0 < 1 < u_0$$

the stock price of S_1 at t_1 is $S_1 = S_0 u_0$ (↑)
$S_1 = S_0 d_0$ (↓)

$\Omega_0 = \{up, down\}$

$\mathcal{F}_1 = P(\Omega_0)$

P_0 → Real world probability (observed probability)
→ theoretical probability (risk neutral probability)

$$P_0(\{\omega\}) = \begin{cases} p_0 ; & \omega = u_0 \\ 1-p_0 ; & \omega = d_0 \end{cases}$$

$0 < p_0 < 1$ in binomial model

$$S_1(\omega) = S_0 \omega \quad \omega \in \Omega_0$$

↑
S_1 is a radom variable

So, one step binomial model has $\{u_0, d_0, p_0\}$ as parameters

No arbitrage conditions:

let $\tilde{r}_0$ ← per period interest rate be the given per-period interest rate
the market is viable iff

$$0 < d_0 < 1(\tilde{r}_0) < u_0$$

$$1(\tilde{r}_0) = \begin{cases} e^{\tilde{r}_0} ; & \text{for continious compounding interest} \\ 1 + \tilde{r}_0 ; & \text{for discrete compounding interest} \end{cases}$$

$S_0 \begin{cases} \nearrow S_0 u_0 = S_1 \\ \searrow S_0 d_0 = S_1 \end{cases}$ $B = S_0(1+\tilde{r}_0)$

let the market be s.t. $I(\tilde{r}) < d_0$ or $u_0 < I(\tilde{r})$

$I(\tilde{r}) < d_0$ → buy stock

$-S_0 I(\tilde{r}) + S_0 d_0 > 0$

$+S_0 u_0 > 0$

$u_0 < I(\tilde{r})$ → sell stock

$S_0 I(\tilde{r}) - S_0 u_0 > 0$

$-S_0 d_0 > 0$

also, if $0 < d_0 < I(\tilde{r}) < u_0$, then if we show this implies market is arbitrage free, then we are done

as if it allows arbitrage, we only have two trades/portfolio

long stock, short bank → Not arbitrage

short stock, long bank → Not arbitrage

so, $\mathbb{P}(V > 0) = 1$ will violate none

Fair game criterion:

The expected pv should be equal to current stock price

$$\mathbb{E}[S_1] = S_0 u_0 p_0 + S_0 d_0 (1 - p_0)$$

$$\text{expected PV} = \frac{\mathbb{E}[S_1]}{I(\tilde{r}_0)}$$

fair game criteria $\Rightarrow \frac{\mathbb{E}[S_1]}{I(\tilde{r}_0)} = S_0$

$\Rightarrow S_0 I(\tilde{r}_0) = S_0 u_0 p_0 + S_0 d_0 (1 - p_0)$

$\Rightarrow (u_0 - d_0) p_0 = I(\tilde{r}_0) - d_0$

$\Rightarrow p_0 = \frac{I(\tilde{r}_0) - d_0}{u_0 - d_0}$, p_0 is called risk-neutral probability

1-step binomial option pricing model:

$\pi_1 = \{t_0 = 0, t_1 = T\}$

H_0 = fair price

let's think of this as a writer, i.e. how much should writer have in hand at t_0 to meet obligation of selling option contract at expiration T

Heading

$$H_T(\omega) = \begin{cases} \max(0, S_0 \omega - k) & \text{; call option} \\ \max(0, k - S_0 \omega) & \text{; put option} \end{cases} \quad \omega = u, d$$

$\pi_1 = (\phi_1, \theta_1)$ ← Replicating portfolio

$V_T(\pi_1)(T) = H_T$ ← option

↳ spot market portfolio

$\phi_1 B_0 I(\tilde{r}_0) + \theta_1 S_0 d_0 = H_T(d_0)$

$\phi_1 B_0 I(\tilde{r}_0) + \theta_1 S_0 u_0 = H_T(u_0)$

$\pi_1 = (\phi, \theta)$ → stock
↳ risk free

$p = \frac{I(\tilde{r}) - d}{u - d}$

$\phi B_0 I(\tilde{r}) + \theta(S_0 u) = H_T(u)$

$\underbrace{\phi B_0 I(\tilde{r})}$ value of risk free part of our portfolio

$\phi B_0 I(\tilde{r}) + \theta(S_0 d) = H_T(d)$

wog let $B_0 = 1$ ($\because \phi$ now is the money)

Note: $B_0, I(\tilde{r}), S_0, u_0, d_0$ is given to us (parameters of models)

$$\begin{bmatrix} B_0 I(\tilde{r}) & S_0 u_0 \\ B_0 I(\tilde{r}) & S_0 d_0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \end{bmatrix} = \begin{bmatrix} H_T(u_0) \\ H_T(d_0) \end{bmatrix}$$

as $d_0 < u_0 \Rightarrow I(\tilde{r}) S_0 d_0 B_0 < I(\tilde{r}) S_0 u_0 B_0$

$$\Rightarrow \det \begin{bmatrix} B_0 I(\tilde{r}) & S_0 u_0 \\ B_0 I(\tilde{r}) & S_0 d_0 \end{bmatrix} \neq 0$$

$\Rightarrow$ unique solution (ϕ, θ)

so, in arbitrage free space $\exists$ unique solution

$$\Rightarrow \phi = \frac{u H_T(d) - d H_T(u)}{B I(\tilde{r})(u - d)} \qquad \theta = \frac{H_T(u) - H_T(d)}{S_0(u_0 - d_0)}$$

so, writer will wage atleast intial value, buyer will pay almost initial value (law of one price)

so, $V_0 = \phi B_0 + \theta S_0$ = value of portfolio π_1 at $t_0 = 0$

$= H_0$

$$H_0 = V_0 \Rightarrow H_0 = \left(\frac{u_0 H_T(d_0) - d_0 H_T(u_0)}{I(\tilde{r})(u_0 - d_0)} \right) + \left(\frac{H_T(u_0) - H_T(d_0)}{(u_0 - d_0)} \right)$$

$$= \frac{H_T(u_0)}{I(\tilde{r})} \left[\frac{I(\tilde{r})}{u_0 - d_0} - \frac{d_0}{u_0 - d_0} \right]$$

$$+ \frac{H_T(d_0)}{I(\tilde{r})} \left[\frac{u_0}{u_0 - d_0} - \frac{I(\tilde{r})}{u_0 - d_0} \right]$$

$$= \frac{H_T(u_0)}{I(\tilde{r})} p + \frac{H_T(d_0)}{I(\tilde{r})} (1 - p)$$

↳ Risk neutral probability

Note: $H_0 = \frac{\mathbb{E}[H_T]}{I(T)}$ discounted expected value of payoff under risk neutral measure

6th March:

$H_0 = \mathbb{E}^*[H_T^*]$ ← discounted payoff

↑ Risk neutral measure

$H^* = \frac{H}{I(\tilde{r}_0)}$ $\tilde{r}_0$ = per period interest rate

I = cont / discrete compounding

$\mathbb{E}^* \rightarrow p^* = \begin{cases} p^* = \frac{I(\tilde{r}_0) - d_0}{u_0 - d_0} & \text{; for up} \\ 1 - p^* = \frac{u_0 - I(\tilde{r}_0)}{u_0 - d_0} & \text{; for down} \end{cases}$

for call option

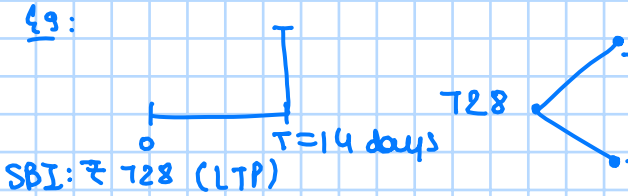
$$C_0^e = \frac{C_T^{e,u} p^*}{I(\tilde{r}_0)} + \frac{C_T^{e,d}(1-p^*)}{I(\tilde{r}_0)}$$

$$= \frac{C_T^{e,u}}{I(\tilde{r}_0)} \left(\frac{I(\tilde{r}_0) - d_0}{u_0 - d_0} \right) + \frac{C^{e,d}}{I(\tilde{r}_0)} \left(\frac{u_0 - I(\tilde{r}_0)}{u_0 - d_0} \right)$$

similarly for put option

so, we need inputs as $\tilde{r}_0, u_0, d_0$

Eg:



fair price of put option

SBI: ₹ 728 (LTP)

per period interest rate = 0.0026 with discrete compounding

$6.5\% = r$

$\frac{r}{365} \times 14 = 0.0026$

$u = \frac{750}{728}$ $d = \frac{700}{728}$

He buys K = 730 → price of this put option is

$$P_0^e = \frac{P_T^{e,u}}{1+r} \left(\frac{1+r-d}{u-d} \right) + \frac{P_T^{e,d}}{(1+r)} \left(\frac{u-(1+r)}{u-d} \right)$$

$$= 0 + \frac{(730-700)}{1+0.0026} \times \frac{1.030219781 - 1.0026}{1.030219781 - \frac{700}{728}} \approx 12$$

Eg: $\tilde{r}_0 = \frac{6.65}{365} \times 4$

$S_0 = 24624$

$K = 24600$

$S_u = 24700$

$S_d = 24500$

$$\frac{(24600-24500)}{1+\frac{6.65 \times 4}{365 \times 100}} \times \left(\frac{\frac{24700}{24624} - 1 - \frac{6.65 \times 4}{365 \times 100}}{\frac{24700}{24624} - \frac{24500}{24624}} \right) \approx 29$$

later we will derive $u = e^{\sigma\sqrt{\Delta t}}$ $d = e^{-\sigma\sqrt{\Delta t}}$

where $\Delta t = \frac{t \text{ days}}{365}$

IV = implied volatility from BSM

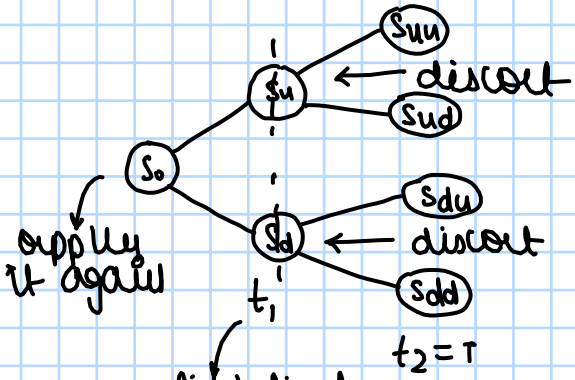
22.01% $\Rightarrow \sigma = 0.2201$

but previously $\sigma = \frac{-1}{\sqrt{\Delta t}} \ln(d)$

$= 0.047 \neq 0.2201$

multi-step Binomial model:

starting from $t_0 = 0$ when stock is purchased for S_0 per share



first find values using one-step binomial model

$\{t_0 = 0, t_1 = \frac{1T}{n}, \ldots t_n = \frac{nT}{n}\}$

given S_{k-1} at $t = t_{k-1}$ for $k = 1, 2, \ldots n$

$S_k = S_{k-1} \times \{u_k, d_k\}$

$\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$

$\Omega_i = \{u_i, d_i\} \quad \forall i \in \{1, 2, \ldots n\}$

$\omega = (\omega_1, \omega_2, \ldots \omega_n)$

$k \in \{1, 2, \ldots n\}$

$\Omega'_k = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_k$ ↑ past to present

$\Omega''_k = \Omega_{k+1} \times \Omega_{k+2} \times \cdots \times \Omega_n$ ↑ present to future

$\Omega = \Omega'_k \times \Omega''_k$

let $\omega = (\omega'_k, \omega''_k)$ $\in \Omega'_k$ $\in \Omega''_k$

and $\omega'_k \in \Omega'_k$ let $B_{\omega'_k} := \{(\omega'_k, \omega''_k) \in \Omega \mid \omega''_k \in \Omega''_k\}$

$P_k = \{B_{\omega'_k} \mid \omega'_k \in \Omega'_k\}$ is partition of Ω

Efficient market hypothesis:

$(\Omega, \mathcal{H}, \mathbb{P})$ $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \times \cdots \times \mathbb{P}_n$, $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ ← every one it's independent

efficient market hypothesis says that current price of risky asset always reflects all available information and changes only with arrival of new information

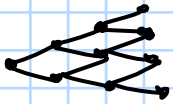
this motivates that stock at $t = t_k$ to be independent of outcomes at all prev time steps

$$\mathbb{P}(\{\omega\}) = \prod_{k=1}^{n} \mathbb{P}(\{\omega_k\}) \quad \omega = (\omega_1, \omega_2, \ldots \omega_n) \in \Omega$$

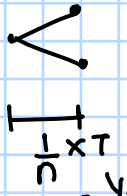
$$\mathbb{P}(\{S_n = S_0 u^j d^{n-j}\}) = \binom{n}{j} p^j (1-p)^{n-j}$$

where $p = \dfrac{I(\tilde{r})^{1/n} - d}{u - d}$ by Fatmans $S_0 = \dfrac{\mathbb{E}[S_T]}{I(\tilde{r})}$

$d = 1/u$ then



as



$$p = \frac{(I(\tilde{r}))^{1/n} - d}{u - d}$$

$\frac{1}{n} \times T$

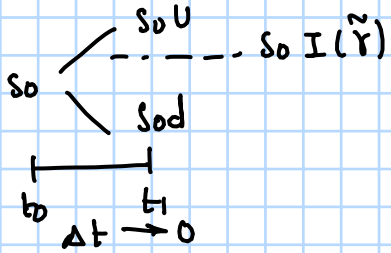
$(I(\tilde{r}))^{1/n}$ ($\because$ continious case)

then we put this p in $\mathbb{P}(\{S_n = S_0 u^j d^{n-j}\})$

$$= \binom{n}{j} (p)^j (1-p)^{n-j}$$

11th March:

we have already seen



$$H_0 = \frac{1}{I(\tilde{r}_0)}\left(H_1^u p^* + H_1^d (1-p^*)\right)$$

$$p^* = \frac{I(\tilde{r}_0) - d}{u-d} \quad \text{(Risk-neutral probability)}$$

Replicable portfolio:

$\pi_1 = (\phi_1, \theta_1)$ (θ_1: delta) $\quad \theta_1 = \frac{H_1^u - H_1^d}{S_0(u-d)} \approx \frac{\partial H}{\partial S} = \Delta$ (delta)

↳ Delta hedge portfolio / replicating portfolio

↑ By solving system of equations

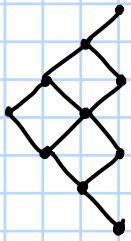
$$(\phi_1, \theta_1) = \left(\frac{uH_1^d - dH_1^u}{I(\tilde{r})(u-d)}, \frac{H_1^u - H_1^d}{S_0(u-d)}\right)$$

N-step binomial model:

payoff $= H_1 = \max(0, S_T - K)$ (long call payout)

we want to construct a portfolio s.t $V_1(\pi_1) = H_1$, then law of one-price tell us $V_0(\pi_1) = H_0$

so, H_0 = price of call option

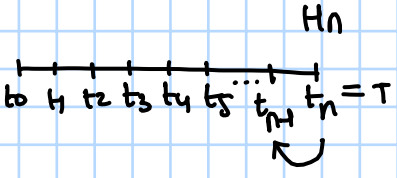


$u = \frac{1}{d}$

constructing a replicating strategy

$\{\pi_k \mid k = 1, 2, \dots n\}$ s.t

$V_n = H_T \quad V_n = V_n(\pi_n)$



π_n replicating portfolio

$$V_n(\pi_n) = H_{t_n}$$

then

$$V_{n-1}(\pi_n) = H_{t_{n-1}}$$

similarly, $V_4(\Pi_4) = H_4$, $V_3(\Pi_4) = H_3$ ($\because$ self-finance)

$$V_3(\Pi_4) = V_3(\Pi_3) = H_3 \rightarrow V_2(\Pi_3) = H_2$$

$$V_2(\Pi_3) = V_2(\Pi_2) = H_2 \rightarrow V_1(\Pi_2) = H_1$$

$$V_1(\Pi_2) = V_1(\Pi_1) = H_1 \rightarrow V_0(\Pi_1) = H_0$$

$$V_0(\Pi_1) = V_0(\Pi_0) = H_0$$

i.e $\phi_n B_{n-1} I(\tilde{\gamma}) + \theta_n S_{n-1} d = H_n^d$

$\phi_n B_{n-1} I(\tilde{\gamma}) + \theta_n S_{n-1} u = H_n^u$

to get $(\phi_n, \theta_n) = \left(\dfrac{u H_n^d - d H_n^u}{B_{n-1} I(\tilde{\gamma})(u-d)} , \dfrac{H_n^u - H_n^d}{S_{n-1}(u-d)} \right)$

this is a predictable process as $S_n = \begin{cases} S_{n-1} u \\ S_{n-1} d \end{cases}$ so at t_{n-1} predictable

also the self-financing condition

$\Rightarrow V_{n-1}(\Pi_n) = \phi_n B_{n-1} + \theta_n S_{n-1}$

$$V_{n-1} = \frac{1}{I(\tilde{\gamma})} \left(H_n^u p^* + H_n^d (1-p^*) \right) \quad \left(V_{n-1} = V_{n-1}(\Pi_{n-1}) = V_{n-1}(\Pi_n) \right)$$

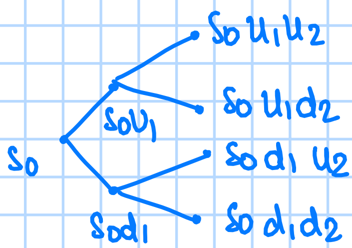
$$V_{n-1} = \mathbb{E}^*[H_n^* \mid \mathcal{F}_{n-1}]$$

$$\mathcal{F}_{n-1} = \sigma(P_{n-1})$$

Theorem: Above replicating portfolio gives $H_t = V_t \ \forall t \in \{0, 1, \dots T\}$

Eg: $\Pi_2 = \{ t_0 = 0, t_1, t_2 = T \}$

$$\Omega = \{(u_1, u_2), (u_1, d_2), (d_1, u_2), (d_1, d_2)\}$$



$P_1 = \{B_U, B_D\}$ $B_U = \{(u_1, u_2), (u_1, d_2)\}$

$B_D = \{(d_1, u_2), (d_1, d_2)\}$

$P_2 = \{B_{UU}, B_{UD}, B_{DU}, B_{DD}\}$ $B_{DD} = \{(UD_1, UD_2)\}$

$$H_2 = \begin{cases} H_T^{UU} & ; B_{UU} \\ H_T^{UD} & ; B_{UD} \\ H_T^{DU} & ; B_{DU} \\ H_T^{DD} & ; B_{DD} \end{cases}$$

$$H_1 = \frac{1}{I(\tilde{r}_2)} \begin{cases} H_2^{uu} p_2^* + H_2^{ud}(1-p_2^*) & ; B_U \\ H_2^{du} p_2^* + H_2^{dd}(1-p_2^*) & ; B_D \end{cases} = \begin{cases} H_1^U & ; B_U \\ H_1^D & ; B_D \end{cases}$$

then $H_0 = \dfrac{1}{I(\tilde{r}_1)} \left(H_1^U p_1^* + H_1^D (1-p_1^*) \right)$

Note: we don't need to make $\{\Pi_i\}$ to calculate $\{H_i\}$, but in market, to trade we need $\{\Pi_i\}$

Explicit form of H_0:

$\mathcal{H}_0$ has no information

$$H_0 = \mathbb{E}^*[H_1^* \mid \mathcal{H}_0] = \mathbb{E}[H_1^*]$$

$$= \frac{1}{I(\tilde{r})}\left(H_1^U p^* + H_1^D (1-p^*)\right)$$

$$= \frac{1}{(I(\tilde{r}))^2}\left(\left(H_2^{UU} p^* + H_2^{UD}(1-p^*)\right)p^* + \left(H_2^{DU} p^* + H_2^{DD}(1-p^*)\right)(1-p^*)\right)$$

$$= \frac{1}{(I(\tilde{r}))^2}\left(H_2^{UU} p^* p^* + H_2^{UD} p^*(1-p^*) + H_2^{DU}(1-p^*)p^* + H_2^{DD}(1-p^*)(1-p^*)\right)$$

$$= \frac{1}{(I(\tilde{r}))^2}\left[\sum_{i=0}^{2} H_{2,i}\binom{2}{i}(1-p^*)^{2-i}(p^*)^i\right]$$

Theorem: (multi-step binomial model) Let $[0,T]$ be partitioned into n subintervals, price of attainable European option in no-arbitrage market is uniquely given by

$$H_0 = \frac{1}{I(\tilde{r})^n}\sum_{j=0}^{n}\binom{n}{j}(p^*)^j(1-p^*)^{n-j}H_{n,j}$$

The binomial model is also called CRR model (Cox-Ross Rubinstein)

Note: in above formula, we assume $d = \frac{1}{u}$, $\tilde{r}_i = r$, Δt same, p^* same, u_i same, u_d same

$u = e^{\sigma\sqrt{\Delta t}}$ σ = implied volatility

$d = e^{-\sigma\sqrt{\Delta t}}$

Lognormal model:

Let $S_t = S_0 e^{R_t}$ R_t is a random variable

$$\left\{t_0 = 0, t_1 = \frac{1}{n}, t_2 = \frac{2}{n}, \dots t_{n-1} = \frac{n-1}{n}, t_n = 1\right\}$$

at $t = t_{k+1}$ as $S_{k+1} = S_k e^{R_{k+1}}$ true at $t_n = 1$:

$$S_n = \frac{S_n}{S_{n-1}} \times \frac{S_{n-1}}{S_{n-2}} \times \cdots \times \frac{S_1}{S_0} S_0$$

$$= S_0 \exp\left\{\sum_{k=1}^{n} R_k\right\}$$

$$\Rightarrow \ln(S_n) = \ln(S_0) + \sum_{k=1}^{n} R_k$$

$$\Rightarrow \ln\left(\frac{S_n}{S_0}\right) = \sum_{k=1}^{n} R_k$$

motivated by Efficient market hypothesis $R_k \overset{iid}{\sim} f(x|\theta)$

Let R_k have $\mu < \infty$, $\sigma^2 < \infty$, μ, σ are annualized mean and SD of log return

$$\mathbb{E}\left[\ln\left(\frac{S_n}{S_0}\right)\right] = \mu T = n\tilde{\mu}$$

$$\text{var}\left[\ln\left(\frac{S_n}{S_0}\right)\right] = \sigma^2 T = n\tilde{\sigma}^2$$

given so, we want to find $S_{\Delta t}$

$$S_n = S_{n-1} e^{R_n}$$

then $S_{\Delta t} = \begin{cases} S_0 u \text{ ; with prob } p \\ S_0 d \text{ ; with prob } 1-p \end{cases}$

$n=1,\ T=\Delta t \Rightarrow \quad p\ln u + (1-p)\ln d = \mu\Delta t \quad \text{——①}$

as $\mathbb{E}\left[\ln\left(\frac{S_1}{S_0}\right)\right] = p\ln u + (1-p)\ln d$

as $\text{var}\left[\ln\left(\frac{S_1}{S_0}\right)\right] = \mathbb{E}\left[\ln\left(\frac{S_1}{S_0}\right)^2\right] - \mu^2\Delta t^2 = p(1-p)(\ln u - \ln d)^2$

similarly, $p(1-p)(\ln u - \ln d)^2 = \sigma^2\Delta t \quad \text{——②}$

also let $u = \frac{1}{d} \quad \text{——③}$

then we have 3 equations and 3 unknowns

$$\Rightarrow \begin{matrix} (2p-1)\ln u = \mu\Delta t \\ 4p(1-p)(\ln u)^2 = \sigma^2\Delta t \end{matrix}$$

$$\Rightarrow u = \exp\left\{\sqrt{(\mu\Delta t)^2 + \sigma^2\Delta t}\right\}$$

$$d = \exp\left\{-\sqrt{(\mu\Delta t)^2 + \sigma^2\Delta t}\right\}$$

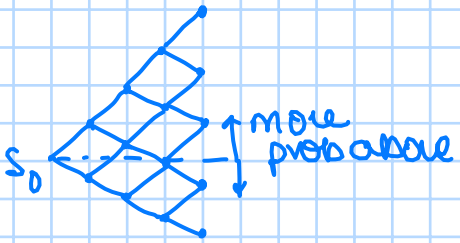
$$p = \frac{1}{2} + \frac{1/2}{\sqrt{\frac{\sigma^2}{\mu^2\Delta t + 1}}}$$

as $\Delta t \to 0$

$$u \approx \exp\{\sigma\sqrt{\Delta t}\}$$ ← more spread

$$d \approx \exp\{-\sigma\sqrt{\Delta t}\}$$ ← less spread

$$p \approx \frac{1}{2} + \frac{1}{2}\left(\frac{|\mu|}{\sigma}\right)\sqrt{\Delta t}$$ ← as up spread more, $p > \frac{1}{2}$



13th March:

we have seen the lognormal model $n=1, T=\Delta t$

$$p\ln(u) + (1-p)\ln d = \mu\Delta t$$

$$p(1-p)(\ln(u)-\ln(d))^2 = \sigma^2\Delta t$$

$$u = \frac{1}{d}$$

this 3 equations give us $u \approx e^{\sigma\sqrt{\Delta t}}$, $d \approx e^{-\sigma\sqrt{\Delta t}}$

$$p \approx \frac{1}{2} + \frac{|\mu|}{2\sigma}\sqrt{\Delta t}$$

so far we have seen American and European option.

↳ exercised at any time $t \in [0,T]$ ↳ exercised only at $t=T$

$H^e \leq H^a$ we have seen this

also, for call option: $H^e = H^a$

American options:

$\Pi_n = \{t_0=0, t_1, \dots t_n=T\}$

$(\Omega, \mathcal{H}, \mathbb{P}, \{\mathcal{H}_k\}_{k=0}^n)$ where price process is adapted

exercise at any time t_i

Def: (stopping time) A random variable $\tau: \Omega \to \bar{\mathbb{N}}$ is a stopping time wrt the filtration $(\mathcal{H}_k)$ if

$$\{\tau \leq k\} \in \mathcal{H}_k \quad k=0,1,2,\dots$$

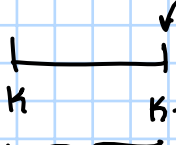
so, τ represents the time at which the holder exercised the option

we need $\underbrace{\{\omega \in \Omega \mid \tau(\omega) \leq k\}}_{\text{is } \mathcal{H}_k \text{ measurable}} \in \mathcal{H}_k$

$H_k = \begin{cases} \max(0, S_k - K) & \text{call} \\ \max(0, K - S_k) & \text{put} \end{cases}$

if we exercise the option

Hold the option until t_{k+1} — we may exercise



↳ European model

so, $V_k = \mathbb{E}^*\left[\frac{H^a_{k+1}}{I(\tau)} \middle| \mathcal{H}_k\right]$

↳ value at t_k for holding it

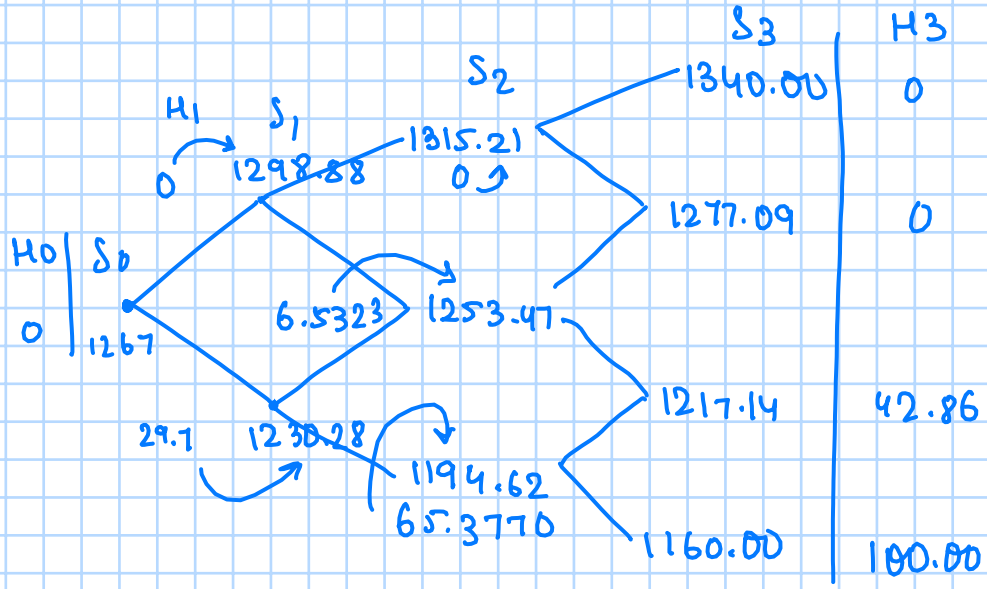
$$H^a_k = \begin{cases} H_n & : k=n \\ \max(H_k, V_k) & : k=0,1,\dots n-1 \end{cases}$$

exercised if $H_k \geq V_k$
continue if $H_k < V_k$

optimal stopping time is $T_*(\omega) = \min\{t_k \in T_n \mid H_k(\omega) \geq V_k(\omega)\}$
↳ first time when we exercise option

eg: american put
$S_0 = 1267$
$K = 1260$
$r = 0.00132$
$u = 1.01885$
$d = 0.971018$

$p^* = 0.63344$ $1-p^* = 0.36656$



0
0
0
0
0
65323
29.7206
42.86
65.3770
100

$V_k = \mathbb{E}^*\left(\frac{H^*}{I(\tilde{r})} \middle| \mathcal{F}_k\right)$ $V_k = \frac{1}{1+\tilde{r}}\left(H_{k+1}^{q,u} p^* + H_{k+1}^{q,d}(1-p^*)\right)$

V_k: V_2 → H_2^q → V_1 → H_1^q → V_0 → H_0^q

0
0
5.744
5.744
16.03
16.03
15.69
15.69
33.859
33.859
63.72
65.3770

(0,0)
(0,0)
(5.744, 0)
(0,0)
(0,0)
(16.03, 0)
(15.69, 6.5323)
(0, 42.86)
(33.859, 29.7206)
(63.72, 65.3770)
(0, 100)

S
S S
C S
C C S
C S
S S

$\tau^*(u,u,u) = 2$

$\tau^*(u,u,d) = 2$

$\tau^*(u,d,u) = 3$

$\tau^*(u,d,d) = 3$

$\tau^*(d,u,u) = 3$

$\tau^*(d,u,d) = 3$

$\tau^*(d,d,u) = 2$

$\tau^*(d,d,d) = 2$

Exotic option:

Any option which depends on the path

$$H_T = H_T(S_1, S_2, \ldots S_T)$$

as till now $H_T = \max(0, S_T - K)$ or $\max(0, K - S_T)$

Asian option is one such type

- fixed strike $\bar{S}$ — some sort of average of price
- floating strike $\bar{K}$ — some sort of floating strike

$H_T = \max(0, S_T - A(S))$ or $\max(0, A(S) - S_T)$

$$A_S = \frac{1}{n}\sum_{k=1}^{n} S_k$$

or $$A_S = \frac{1}{T}\int_0^T S(t)\,dt$$

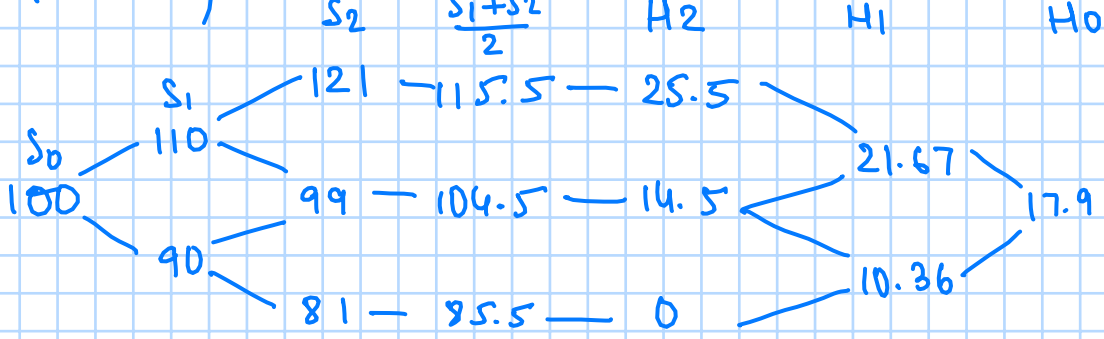
or even geometric averages

Eg: $H_2 = \max\left(0, \frac{1}{2}(S_1 + S_2) - K\right)$

$S_0 = 100$
$K = 90$

$u = 1.1$
$d = 0.9$
$r = 0.1 \Rightarrow \tilde{r} = 0.05$

$p^* = 0.75$
$1 - p^* = 0.25$



18th March:

Marginfale theory:

Normalized finanial market, conditional expectetation, Discrete Marginfale, Fundamentaltheorems of asset pricing

so far we make a replicating portfolio, we will make a portfolio

price is determined by the cost of hedging portfolio

so, equivalent representation of option price = Exitance of risk-neutral probability measure

$$H_0 = H_1^* p^* + H_1^* (1-p^*) = \mathbb{E}^*[H_1^*]$$

Note: we need risk neutral portfolio

Def: (complete market) if every contingent claim can be replicated by a self finaling strategy

(B, S) if $S \in \mathbb{R}^m$ (m stocks), then $H \to 2^m$ portfolios
but $1+m$ variables
$\Rightarrow m+1$ - model (not binomial model)

but ① the system may not have solution
② the system has infinite many solutions

so, we have a market, we want unique replicating portfolio

(B, S) → ① market is complete (portfolio exist) (p, θ exist)
② measure is playing a role in fair price of model

$Ax = b$, A = price process
x = portfolio
b = payoff

existance of portfolio if solution exist

NO arbitrage is giving risk neutral probability

$$p^* = \frac{I(\tilde{r}) - d}{u - d} \in [0,1]$$ if probability

$\in [0,1]$ comes from NO arbitrage condition

$\uparrow$ any exist >0 in finance

$$d < I(\tilde{r}) < u$$

if it exist then, $V = \phi B + \theta S$, so if NO-arbitrage

Binomial ($S \in \mathbb{R}$) model
if arbitrage free → p^* exist
$\Rightarrow p^*$ unique and V_0 unique

we will find iff condition for uniqueness in genral framework

Neural model:

Binomial Model	Neural Direct-Time Market
- Replicating portfolio exist	- market is complete
- P^* Risk neutral probability	- equivalent martingale measure P^*
- Risk-Neutral Pricing formula	- martingale pricing formula

Normalised financial market:

assume $B_t > 0 \longrightarrow 1$

i.e. $(B, S) \to (1, \tilde{S})$ $\quad \tilde{S} = \frac{S}{B}$ ← dimensionless quantity

Normalized finance market or discounted market means

$$\tilde{S}_k = \frac{S_k}{B_k} \quad k = 0, 1, \ldots n$$

$\{B_k\}$ is called Numeraire

$\tilde{V}_k = \phi_k + \theta_k \cdot \tilde{S}_k \quad k = 1, 2, \ldots n$ Π_k at $t = t_k$ ← is the discounted value

$\{\tilde{V}_k\}$ is called the discounted value process

Lemma: (Linear property) Let X_1 and X_2 be random variables on $(\Omega, \mathcal{H}, \mathbb{P})$, for any $a, b \in \mathbb{R}$

$$\mathbb{E}[aX_1 + bX_2 | \mathcal{H}_1] = a\,\mathbb{E}[X_1 | \mathcal{H}_1] + b\,\mathbb{E}[X_2 | \mathcal{H}_2]$$

$\mathcal{H}_1 \subseteq \mathcal{H}$ is σ-field on Ω

Def: (Discrete Martingale) $\{M_k\}$ on $(\Omega, \mathcal{H}^*, \mathbb{P})$ is martingale wrt $\{\mathcal{H}_k\}$ if

① $\{M_k\}$ is adapted to $\{\mathcal{H}_k\}$

② $\forall k = 0, 1 \ldots n-1$

$\mathbb{E}[|M_k|] < \infty$ and $\mathbb{E}[M_{k+1} | \mathcal{H}_k] = M_k$ a.s

we say that $\{M_k\}$ is $\mathcal{H}_k$-martingale

Note: we also have supermartingale ($\leq$)
submartingale ($\geq$)

Fair game criterion:

$\{M_k\}$ → winning amount of gambler at different time intervals

$\Rightarrow M_{k+1} - M_k$ gain accumulation during period $(t_k, t_{k+1}]$

then $\mathbb{E}[M_{k+1} - M_k | \mathcal{H}_k] = \mathbb{E}[M_{k+1} | \mathcal{H}_k] - \mathbb{E}[M_k | \mathcal{H}_k]$

$= 0 \quad (\because \mathbb{E}[M_k | \mathcal{H}_k] = M_k)$

$(\because$ linearity$)$

Lemma: $\{\tilde{S}_k\}$ from multi-step binomial model is an arbitrage free market with risk-neutral probability is $\mathcal{H}_k$-martingale wrt $\{\mathcal{H}_k\}$ s.t $\mathcal{H}_k^S$ = Natural/generated σ-field

$\mathcal{H}_k^S \subseteq \mathcal{H}_k$ for $k = 1, 2 \ldots n$

Lemma: Let $\{\pi_k\}$ be a self-financing strategy and let $\{\tilde{V}_k\}$ be the corresponding discounted value process, then $\{\tilde{V}_k\}$ is $\mathcal{H}_k$-martingale in the risk neutral prob space with $\{\mathcal{H}_k\}$ s.t $\mathcal{H}_k^S \subseteq \mathcal{H}_k$ $\forall k = 1, 1 \ldots n$

(B, S) where $S = (S^1, S^2, \ldots S^m)$ be defined on a filtred probability space $(\Omega, \mathcal{H}^*, \mathbb{P}, \{\mathcal{H}_k\})$

Def: Two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on $(\Omega, \mathcal{F})$ are equivalent (or mutually absolute continuous) if $\forall A \in \mathcal{F}$

$$\mathbb{P}_1(A) = 0 \text{ iff } \mathbb{P}_2(A) = 0$$

20th March:

Generalizing the binomial model:

Binomial model	General Discrete-Time Market
- Replicating portfolio exist	- Market is complete
- Risk-Neutral prob P^*	- equivalent martingale measure P^*
- Risk-Neutral pricing formula	- martingale pricing formula

assume $B_t > 0$

Normalised finacial market or discounted market means

$$\tilde{S}_k = \frac{S_k}{B_k}, \quad k = 0,1,2 \dots n$$

the proces $\{B_k\}$ is called a numeraire

in a discounted market, value of a portfolio Π_k at $t = t_k$ is given by

$$\tilde{V}_k = \phi_k + \theta_k \cdot \tilde{S}_k \quad k = 1,2,\dots n$$

and is called the discounted value

$\{\tilde{V}_k\}$ is called the discounted value proces

Note: $V_0 = V_0(\Pi_1)$

Lemma: Strategy $\{(\phi_k, \theta_k) \mid k = 0,1,2,\dots n\}$ is self-financing iff

$$\phi_{k+1} = \tilde{V}_0 - \sum_{j=0}^{k} \Delta\theta_j \cdot \tilde{S}_j \quad k = 0,1 \dots n-1$$

where $\vec{\theta}_0 = \vec{0}$

proof:

we used $\Delta\phi_k B_k + \Delta\theta_k \cdot S_k = 0$ for self financing

$k = 1,2 \dots n-1$

$$\Delta\phi_k = -\Delta\theta_k \cdot \tilde{S}_k$$

$$\phi_{k+1} - \phi_k = -\Delta\theta_k \cdot \tilde{S}_k$$

$$\phi_k - \phi_{k-1} = -\Delta\theta_{k-1} \tilde{S}_{k-1}$$

$$\vdots$$

$$\phi_1 - \phi_0 = -\Delta\theta_0 \tilde{S}_1$$

$$\phi_0 + \theta_0 \tilde{S}_0 = \tilde{V}_0 \quad (\because \vec{\theta}_0 = \vec{0})$$

$$\Rightarrow \phi_{k+1} = \tilde{V}_0 - \sum_{j=0}^{k} \Delta\theta_j \cdot \tilde{S}_j \quad k = 0,1,2 \dots n-1$$

Defn: (conditional expectation) Let X be a r.v on $(\Omega, \mathcal{H}, P)$ with $\mathbb{E}[|X|] < \infty$ and let $\mathcal{H}_1 \subseteq \mathcal{H}$ be a σ-field then a $\mathcal{H}_1$-measurable r.v Y on $(\Omega, \mathcal{H}, P)$ with

$\mathbb{E}[|Y|]<\infty$ is said to be a conditional expectation of X given $\mathcal{H}_1$, denoted by

$Y = \mathbb{E}[X|\mathcal{H}_1]$ iff $\mathbb{E}[Y \mathbb{I}_A] = \mathbb{E}[X \mathbb{I}_A] \quad \forall A \in \mathcal{H}_1$

where $\mathbb{I}_A$ is the indicator fution of the set A

The following follows from Radon-Nikodym Theorm

Y always exist and is unique $\mathbb{P}$-a.s
Y is $\mathcal{H}_1$-measurable

eg: $(\Omega, \mathcal{H}^*, \mathbb{P}, \{\mathcal{H}_k\})$ on $\Pi_n = \{t_0, t_1 \dots t_n\}$

for $\omega \in \Omega$ let $\mathbb{P}(\omega) = p^{*U(\omega)}(1-p^*)^{D(\omega)}$ $U \to$ NO of ups

$\Omega = \Omega_1 \times \Omega_2 \dots \times \Omega_n$

$\omega = (\omega_1, \omega_2, \dots \omega_n) \in \Omega$

$\forall k = 1, 2, \dots n$

$\Omega_k = \{up, down\}$ $U(\omega) =$ no of ups, $D(\omega) =$ no. of downs

$\mathcal{H}_n = \sigma(\mathcal{P}_k)$ ← collection of sets B_{ω_k} given by

$B_{\omega_k} = \{\omega \in \Omega \mid$ first k components equal to $\omega_k\}$

σ-field $\mathcal{H}_k$ represents information generated by moment of stock at first k time intervals

for any given $k \in \{1, 2, \dots n\}$

$\Omega_k' = \Omega_1 \times \dots \times \Omega_k$

$\Omega_k'' = \Omega_{k+1} \times \dots \times \Omega_n$

let $\Omega = \Omega_k' \times \Omega_k''$ $\forall k = 1, 2 \dots n$ any $\omega \in \Omega$

$\omega = (\omega_k', \omega_k'')$

$B_{\omega_k'} = \{(\omega_k', \omega_k'') \mid \omega_k'' \in \Omega_k''\}$

$\mathbb{E}[X|\mathcal{H}_k](\omega) = \sum_{\alpha \in \Omega_k''} p^{*U(\alpha)}(1-p^*)^{D(\alpha)} X(\omega_k', \alpha)$

$= Y(\omega) = Y(\omega_k')$

only fution of ω_k' so, Y is $\mathcal{H}_k$-measurable

if we show $\mathbb{E}[Y \mathbb{I}_{B_{\omega_k'}}] = \mathbb{E}[X \mathbb{I}_{B_{\omega_k'}}]$ we are done (∵ generators)

now, $\mathbb{E}[Y \mathbb{I}_{B_{\omega_k'}}] = \sum_{\omega \in B_{\omega_k'}} Y(\omega)\mathbb{P}(\omega) = Y(\omega_k') \sum_{\omega \in B_{\omega_k'}} \mathbb{P}(\omega)$

$= \left(p^{*U(\omega_k')}(1-p^*)^{D(\omega_k')}\right)\left(\sum_{\alpha \in \Omega_k''} p^{*U(\alpha)}(1-p)^{D(\alpha)} X(\omega_k', \alpha)\right)$

$\left(\underbrace{\sum_{\alpha \in \Omega_k''} p^{*U(\alpha)}(1-p^*)^{D(\alpha)}}_{1}\right)$

$= \sum_{\omega \in B_k'} X(\omega)\mathbb{P}(\omega)$

$= \mathbb{E}[X \amalg_{B\omega_k}]$

so, $Y(\omega)$ is conditional expectation of X given $\mathcal{H}_1$

Exe: show the linearity property

X_1, X_2 r.v $(\Omega, \mathcal{H}, \mathbb{P})$, for any $a, b \in \mathbb{R}$

show that $\mathbb{E}[aX_1 + bX_2 | \mathcal{H}_1] = a\mathbb{E}[X_1|\mathcal{H}] + b\mathbb{E}[X_2|\mathcal{H}_1]$

$\mathcal{H}_1 \subseteq \mathcal{H}$ is a σ-field on Ω

Ans:

let $Y_1 = \mathbb{E}[X_1 | \mathcal{H}_1] \Rightarrow \mathbb{E}[Y_1 \amalg_A] = \mathbb{E}[X_1 \amalg_A] \quad \forall A \in \mathcal{H}_1$

$Y_2 = \mathbb{E}[X_2 | \mathcal{H}_2] \Rightarrow \mathbb{E}[Y_2 \amalg_A] = \mathbb{E}[X_2 \amalg_A] \quad \forall A \in \mathcal{H}_1$

$\tilde{Y} = aY_1 + bY_2 \qquad X = aX_1 + bX_2$

then $\mathbb{E}[\tilde{Y} \amalg_A] = \mathbb{E}[(aY_1 + bY_2) \amalg_A]$

$= a\mathbb{E}[Y_1 \amalg_A] + b\mathbb{E}[Y_2 \amalg_A]$

$= \mathbb{E}[X \amalg_A]$

$\Rightarrow \tilde{Y} = \mathbb{E}[X | \mathcal{H}_1]$

Exe: show the tower property i.e let X be r.v on $(\Omega, \mathcal{H}, \mathbb{P})$ and let $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \mathcal{H}$, show that

$\mathbb{E}[\mathbb{E}[X|\mathcal{H}_2] | \mathcal{H}_1] = \mathbb{E}[X|\mathcal{H}_1]$

Ans:

let $Y_2 = \mathbb{E}[X|\mathcal{H}_2]$

$Y_1 = \mathbb{E}[X|\mathcal{H}_1]$

$\mathbb{E}[Y_2 | \mathcal{H}_1] = Y_1$

now, $\forall A \in \mathcal{H}_1, \mathbb{E}[Y_1 \amalg_A] = \mathbb{E}[X \amalg_A] \quad \forall A \in \mathcal{H}_1$

$\Rightarrow A \in \mathcal{H}_2$ ($\mathcal{H}_1 \subseteq \mathcal{H}_2$)

$= \mathbb{E}[X \amalg_A]$ s.t $A \in \mathcal{H}_2$

$= \mathbb{E}[Y_2 \amalg_A]$

$\Rightarrow \mathbb{E}[Y_2 | \mathcal{H}_1] = Y_1$

$\Rightarrow \mathbb{E}[\mathbb{E}[X|\mathcal{H}_2] | \mathcal{H}_1] = \mathbb{E}[X|\mathcal{H}_1]$

Lemma: (Factor property) let X be a r.v on $(\Omega, \mathcal{G}, \mathbb{P})$ and Y be a r.v on $(\Omega, \mathcal{H}, \mathbb{P})$, then show that

$\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$

proof: as $\mathbb{E}[Y|\mathcal{G}]$ is $\mathcal{G}$ measurable r.v, $X\mathbb{E}[Y|\mathcal{G}]$ is also a $\mathcal{G}$ measurable r.v

so, if we show that

$\mathbb{E}[X(\mathbb{E}[Y|\mathcal{G}]) \amalg_A] = \mathbb{E}[XY \amalg_A] \quad \forall A \in \mathcal{G}$

as by def^n $\Leftrightarrow X\,\mathbb{E}[Y|\mathcal{G}] = \mathbb{E}[XY|\mathcal{G}]$

for a special case, when range of X is finite let $\{x_1, x_2, \dots x_n\}$

$A_j = \{X = x_j\}$ then $A_j \in \mathcal{G}$ and

$$X\,\mathbb{I}_A = \sum_{j=1}^{n} x_j \mathbb{I}_{A \cap A_j}$$

$$X\,\mathbb{E}[Y|\mathcal{G}]\,\mathbb{I}_A = \mathbb{E}[Y|\mathcal{G}] \sum_{j=1}^{n} x_j \mathbb{I}_{A \cap A_j}$$

$$\mathbb{E}[X\,\mathbb{E}[Y|\mathcal{G}]\,\mathbb{I}_A] = \mathbb{E}[Y|\mathcal{G}]\left(\sum_{j=1}^{n} x_j \mathbb{I}_{A \cap A_j}\right)$$

$$= \mathbb{E}\left[\sum_{j=1}^{n} x_j Y \mathbb{I}_{A \cap A_j}\right]$$

$(\because$ linearity$)$

$$= \mathbb{E}[XY\,\mathbb{I}_A]$$

lemma: let $\{M_k\}$ be a $\mathcal{H}_k$-martingale, for any $j \le k$

$$\mathbb{E}[M_k|\mathcal{H}_j] = M_j$$

proof: $j \le k-1 \Rightarrow \mathcal{H}_j \subseteq \mathcal{H}_{k-1}$

$$\mathbb{E}[M_k|\mathcal{H}_j] = \mathbb{E}[\mathbb{E}[M_k|\mathcal{H}_{k-1}]|\mathcal{H}_j]$$

$$= \mathbb{E}[M_{k-1}|\mathcal{H}_j] \quad (\because \text{martingale})$$

by induction.

$$\mathbb{E}[M_k|\mathcal{H}_j] = \mathbb{E}[M_{j+1}|\mathcal{H}_j] = M_j$$

lemma: the discounted stock price process $\{\tilde{S}_k\}$ obtained from multi-step binomial model in an arbitrage free market with risk-neutral probability is a $\mathcal{H}_k$-martingale with respect to filtration $\{\mathcal{H}_k\}$ s.t

$$\mathcal{F}_k^S \subseteq \mathcal{H}_k \quad \forall\, k = 1, 2, \dots n$$

lemma: let $\{\tilde{S}_k\}$ be as above, then let $\{\pi_k\}$ be a self-financing strategy and let $\{V_k\}$ be the corresponding discounted value process, then

$\{\tilde{V}_k\}$ is $\mathcal{H}_k$-martingale in risk neutral probspace with

$\{\mathcal{H}_k\}$ filteration s.t $\mathcal{F}_k^S \subseteq \mathcal{H}_k \quad \forall k = 1, 2, \dots n$

lemma: let $\mathbb{P}^*$ be an EMM and $\{\pi_k\}$ be a self-financing strategy in finite market model. Then the discounted value process $\{\tilde{V}_k\}$ is a $\mathcal{H}_k$-martingale wrt $\mathbb{P}^*$

Theorem: (First FTAP) A finite market is viable iff $\exists$ equivalent martingale measure $\mathbb{P}^*$

Defn: (complete market) A market is said to be complete if every contingent claim is replicable

Theorem: (second FTAP) A viable market (B, S) is complete iff $\exists$ unique EMM with numeraire B

25th March:

Discrete martingale:

Defn: $\{M_k\}$ on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_k\})$

① $\{M_k\}$ adapted to $\{\mathcal{F}_k\}$

② $k=0,1,\dots n+1$, $\mathbb{E}[|M_k|]<\infty$ and

$\mathbb{E}[M_{k+1}|\mathcal{F}_k] = M_k$ a.s

Note: $\leq$ is supermartingale

$\geq$ is submartingale

Fair-game:

$\{M_k\}$ → winnings about of a gambler at different time levels

$M_{k+1} - M_k$ gain during $(t_k, t_{k+1}]$

$\{M_k\}$ is adapted ⇒ M_k is $\mathcal{F}_k$-measurable

by linearity $\mathbb{E}[M_{k+1} - M_k | \mathcal{F}_k] = M_k - M_k = 0$

Lemma: the discounted stock price $\{\tilde{S}_k\}$ obtained from multi-step binomial model in arb free with risk neutral prob is $\mathcal{F}_k$-martingale wrt $\{\mathcal{F}_k\}$ s.t $\mathcal{F}_k^S \subseteq \mathcal{F}_k$ $\forall k=1,2,\dots n$

proof: $\{\tilde{S}_k\}$ is adapted $(\because \mathcal{F}_k^S \subseteq \mathcal{F}_k \; \forall k=1,\dots n)$

now, to show $\forall k=0,1\dots n-1$, $\mathbb{E}^*[\tilde{S}_k | \mathcal{F}_{k+1}] = \tilde{S}_{k+1}$ then we are done

$$\mathbb{E}^*\left[\tilde{S}_k | \mathcal{F}_{k+1}\right] = \frac{1}{B_k}\mathbb{E}^*\left[S_k | \mathcal{F}_{k+1}\right]$$

$$= \underbrace{\frac{1}{B_0 I(\tilde{r})^k}}_{B_k} \left(S_{k+1} u p^* + S_{k+1} d(1-p^*)\right) \quad (\because \text{Binomial model})$$

$$= \frac{S_{k+1}}{B_0 I(\tilde{r})^k}\left[u\left(\frac{I(\tilde{r})-d}{u-d}\right) + d\left(\frac{u-I(\tilde{r})}{u-d}\right)\right]$$

$$= \frac{S_{k+1}}{B_0 I(\tilde{r})^{k+1}}\left[\frac{u I(\tilde{r}) - ud + du - d I(\tilde{r})}{u-d}\right]$$

$$= \frac{S_{k+1}}{B_0 I(\tilde{r})^{k+1}} = \tilde{S}_{k+1}$$

Lemma: let $\{\tilde{S}_k\}$ be as above, then $\{\pi_k\}$ be self-financing, $\{\tilde{V}_k\}$ is $\mathcal{F}_k$-martingale in risk-neutral probability space with $\{\mathcal{F}_k\}$ s.t $\mathcal{F}_k^S \subseteq \mathcal{F}_k$

proof:

$\pi_k = (\phi_k, \theta_k)$ $V_k = V_k(\pi_k) = \phi_k B_k + \theta_k S_k \Rightarrow \tilde{V}_k = \phi_k + \theta_k \cdot \tilde{S}_k$

we have to show: (1) $\{\tilde{V}_k\}$ is adapted to $\{\mathcal{F}_k\}$

(2) $\mathbb{E}^*[\tilde{V}_k \mid \mathcal{F}_{k-1}] = \tilde{V}_{k-1}$

now, as $\tilde{V}_k = \phi_k + \theta_k \cdot \tilde{S}_k$

$\{\phi_k\}$ is $\{\mathcal{F}_{k-1}\}$ measurable ($\because$ it is predictable)

similarly $\{\theta_k\}$ is $\{\mathcal{F}_{k-1}\}$ measurable

& $\{\tilde{S}_k\}$ is $\{\mathcal{F}_k\}$ measurable

$\Rightarrow \{\tilde{V}_k\}$ is $\{\mathcal{F}_k\}$ measurable ($\because \{\mathcal{F}_{k-1}\}$ measurable $\Rightarrow \{\mathcal{F}_k\}$ measurable)

now, $\mathbb{E}^*[\tilde{V}_k \mid \mathcal{F}_{k-1}] = \mathbb{E}^*[\phi_k + \theta_k \tilde{S}_k \mid \mathcal{F}_{k-1}]$

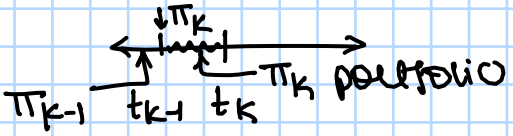
$= \mathbb{E}^*[\phi_k \mid \mathcal{F}_{k-1}] + \mathbb{E}^*[\theta_k \tilde{S}_k \mid \mathcal{F}_{k-1}]$

$= \phi_k + \theta_k \mathbb{E}^*[\tilde{S}_k \mid \mathcal{F}_{k-1}]$ ($\because \theta_k$ is $\mathcal{F}_{k-1}$ measurable, so we use factor property)

$= \phi_k + \theta_k \tilde{S}_{k-1} = \tilde{V}_{k-1}(\pi_k) = \tilde{V}_{k-1}$ ($\because$ self-financing)

($\because \{\tilde{S}_k\}$ is $\{\mathcal{F}_k\}$ measurable)

now $V_{k-1}(\pi_{k-1}) = V_{k-1}(\pi_k)$ ($\because$ self financing)



$\Rightarrow \tilde{V}_{k-1} = \tilde{V}_{k-1}(\pi_k)$

(B, S) be a finite market where $S = (S^1, S^2, \dots S^m)$ on filtered probability space $(\Omega, \mathcal{F}^*, \mathbb{P}, \{\mathcal{F}_k\})$

$H = (h^1, h^2, \dots h^m)$

Def: two prob measures $\mathbb{P}_1, \mathbb{P}_2$ on $(\Omega, \mathcal{F})$ are equivalent if $\forall A \in \mathcal{F}$

$\mathbb{P}_1(A) = 0$ iff $\mathbb{P}_2(A) = 0$

Def: (EMM) given (B, S), $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_k\})$, EMM is prob measure s.t

① $\mathbb{P}^*$ is equivalent to $\mathbb{P}$, $\mathbb{P}^*$ defined on $(\Omega, \mathcal{F}^*)$

② $\{\tilde{S}_k\}$ is $\{\mathcal{F}_k\}$ martingale wrt $\mathbb{P}^*$

Lemma: let $\mathbb{P}^*$ be EMM, $\{\pi_k\}$ self-financing in finite model then $\{\tilde{V}_k\}$ is $\mathcal{F}_k$ martingale wrt $\mathbb{P}^*$

proof: as price process is martingale, it follows similarly to previous lemma

$\{\tilde{S}_k\}$ martingale $\Rightarrow \{\tilde{V}_k\}$ is martingale

Theorm: (First FTAP) A finite market is viable iff $\exists$ EMM $\mathbb{P}^*$

proof: ($\Leftarrow$) EMM exist, now if market is not-viable then

$\exists$ an arbitrage ← only options

(B, S, H)

$\Rightarrow V_0 = 0$,

$\mathbb{P}^*(V_n \geq 0) = 1$

$V_n \geq 0, \mathbb{P}^*$ a.s

& $\mathbb{P}(V_n > 0) > 0$

$\mathbb{E}^*[\tilde{V}_n] = \mathbb{E}^*[\tilde{V}_n \mid \mathcal{F}_0]$

$= \tilde{V}_0$ ($\because \{\tilde{V}_n\}$ is $\{\mathcal{F}_n\}$ martingale)

$= 0$

this is a contradiction as

$\mathbb{E}^*[\tilde{V}_n] > 0$ as $\mathbb{P}[V_n > 0] > 0$ and $V_n \geq 0$ a.s

($\Rightarrow$) market has no arbitrage, then by using hahn-banach geometry theorm

i.e 2 non-intersecting open sets, we can always put a hyper-plane

$$\mathbb{E}^*[\tilde{V}_n] = \mathbb{E}^*[\tilde{V}_n \mid \mathcal{F}_0] = \sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}) \tilde{V}_n(\omega) = 0$$

$\Omega = \Omega^+ \cup \Omega^-$

$\tilde{V}_n(\omega) > 0$

then we will get a contradiction

Defn: A market is said to be complete if every contingent claim is replicable

Theorm: (Second FTAP) A viable market (B, S) is complete iff $\exists$ unique EMM with numeraire B

proof: ($\Rightarrow$) viability and complete, then by first FTAP

$\exists$ EMM $\mathbb{P}^*$, to show now uniqueness, if $\exists \mathbb{P}^{**}$ which is another EMM

$\mathbb{P}^*(A) = \mathbb{P}^{**}(A)$ $\forall A \in \mathcal{F}$ then we are done

27th march:

Theorm: (second FTAP) A viable market (B,S) is complete iff $\exists$ unique EMM with numeraire B

proof: assume that market is viable and complete, any contigent claim can be replicated

then we want to show $\exists$ unique EMM say $\mathbb{P}^*$

from first FTAP, $\exists$ EMM

to show uniqueness, let's assume that $\exists \mathbb{P}^*, \mathbb{P}^{**}$ EMM then if

$$\forall A \in \mathcal{H}, \quad \mathbb{P}^*(A) = \mathbb{P}^{**}(A)$$

we are done ← this can be any contigent claim, not only options

let H be a payoff of a contigent claim

let $\{\pi_k\}$ be a self financing strategy with discounted value process $\{\tilde{V}_k\}$, now completion

$$\Rightarrow \tilde{H}_n = \tilde{V}_0 + \sum_{j=0}^{n-1} \theta_j^0 \cdot \Delta \tilde{S}_j^0 \quad \text{—— ①}$$

$$\Delta \tilde{S}_j^0 = \Delta S_j^0 - \Delta S_{j-1}^0$$

$$\mathbb{E}^*[\theta_j^0 \cdot \Delta \tilde{S}_{j-1}^0 \mid \mathcal{H}_{j-1}] = \theta_j^0 \cdot \mathbb{E}^*[\Delta \tilde{S}_{j-1}^0 \mid \mathcal{H}_{j-1}] = 0$$

($\because$ factor property and martingale)

$$\Rightarrow \mathbb{E}^*[\theta_j^0 \cdot \Delta S_{j-1}^0] = \mathbb{E}^*[\theta_j^0 \cdot \Delta S_{j-1}^0 \mid \mathcal{H}_0]$$

now $\mathbb{E}^*\left[\mathbb{E}^*[\theta_j^0 \cdot \Delta S_{j-1}^0 \mid \mathcal{H}_{j-1}^0] \mid \mathcal{H}_0\right] = \mathbb{E}^*[\theta_j^0 \cdot \Delta \tilde{S}_{j-1}^0 \mid \mathcal{H}_0]$

$$\Rightarrow 0 = \mathbb{E}^*[\theta_j^0 \cdot \Delta S_{j-1}^0 \mid \mathcal{H}_0]$$

$$\Rightarrow \mathbb{E}^*[\theta_j^0 \cdot \Delta \tilde{S}_{j-1}^0] = 0$$

now, in ①, $\mathbb{E}^*[\tilde{H}_n] = \tilde{V}_0$

following same idea, $\mathbb{E}^{**}[\tilde{H}_n] = \tilde{V}_0$

in particular $H_n = \mathbb{I}_A$ ← example contigent claim when event A happens

$$\Rightarrow \mathbb{E}^*[\tilde{H}_n] = \frac{1}{I(\tilde{Y})}\left[1 \times \mathbb{P}^*(A) + 0 \times \mathbb{P}^*(A^c)\right] = \frac{\mathbb{P}^*(A)}{I(\tilde{Y})} \quad \text{—— ②}$$

similarly $\mathbb{E}^{**}[\tilde{H}_n] = \frac{1}{I(\tilde{Y})}\mathbb{P}^{**}(A) \quad \text{—— ③}$

②, ③ $\Rightarrow \mathbb{P}^*(A) = \mathbb{P}^{**}(A) \quad \forall A \in \mathcal{H}$

Exe: show $\tilde{H}_n = \tilde{V}_0 + \sum_{j=0}^{n-1} \theta_j^0 \cdot \Delta \tilde{S}_{j-1}^0$ by induction → done down

Theorm: suppose finite market model is viable and complete, and $\mathbb{P}^*$ is unique EMM, then for any european contingent claim with payoff H, the process

$$\{B_k \mathbb{E}^*[H^* \mid \mathcal{H}_k] \mid k = 0, 1, 2, \ldots n\}$$ is unique arbitrage

how price process for european contingent claim, use

$H^* = \frac{H}{B_n}$ is discounted payoff

we have only done binomial model with (B, S), whether model we do the above theorm hence model does not matter, all we need is unique EMM (viable and complete)

but, real markets are typically incomplete as

① No of source of uncertainty exceeds no of traded assets
② some risks are non-tradable (weather, wait, liquidity)
③ market frictions as trailing cost and trading bust-rails

In a complete market, model give no ambiguity, so all options same priced

further direction: how sound derivatives be priced, when EMM not unique

we this is solved using, utility function (where investor risk)

exponential utility : minimal entropy MM
quadratic utility : variance-optimal MM

or risk-minimization and quadratic hedging (minimising mse)

indifference pricing ← in real-life volatility value, stochastic process
stochastic volatility and incomplete market models

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$
$$d\sigma_t = \text{OU model}$$

Note: we can use deep learning also

$\max_{\Pi} \mathbb{E}[U(X_T^{\Pi})]$ minimise to get parameters

$\Pi_t = f_\theta(S_t, t) \rightarrow f_\theta$ is neural network

network is trained to optimise expected utility, hedging error, risk-measure

continious time option pricing model:

$\Pi_n = \{t_0, t_1, \ldots t_n = T\}$ $\Delta t = t_k - t_{k-1} = \frac{T}{n}$

discrete model only gives us ϕ_n, θ_n, no manipulation/usage between 2

we want $\Delta t \to 0$
the binomial model H_0 formula → some value

$H_0^{(1)}, H_0^{(2)}, \ldots H_0^{(n)}$ $\lim_{n\to\infty} H_0^{(n)}$ will it converge and if so then what is the limit
↑ 1-step ↑ 2-step ↑ n-step

the limit is called black-scholes model

$$S_T^{(1)} = \begin{cases} S_0 u & \text{with } p \\ S_0 d & \text{with } 1-p \end{cases} \qquad S_T^{(2)}, S_T^{(3)}, \ldots \{S_T^{(n)}\}$$

$\lim_{n\to\infty} S_T^{(n)}$ will give back $dS_t = \mu S_t dt + \sigma_t S_t dW_t$ i.e

$$\lim_{n\to\infty} S_T^{(n)} = S_T$$

we have seen from log-normal model

$$\left.\begin{aligned} \mathbb{E}\left[\ln\left(\frac{S_n}{S_0}\right)\right] &= M \\ \operatorname{var}\left[\ln\left(\frac{S_n}{S_0}\right)\right] &= \sigma^2 \end{aligned}\right\} \Rightarrow \begin{aligned} n\tilde{\mu} &= T\mu \\ n\tilde{\sigma}^2 &= T\sigma^2 \end{aligned}$$

← from scaling

$$\text{as } S_t = S_0 e^{R_t} \quad \ln\frac{S_t}{S_0} = \sum_{t=1}^{tn} R_t \to \tilde{\mu}, \tilde{\sigma}^2 \text{ iid}$$

$T=1$:

$$\mathbb{P}\left[\frac{\ln\left(\frac{S_n}{S_0}\right)-\mu}{\sigma} \le r\right] = \mathbb{P}\left[\frac{\ln\left(\frac{S_n}{S_0}\right)-n\tilde{\mu}}{\tilde{\sigma}\sqrt{n}} \le r\right]$$

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\ln\left(\frac{S_n}{S_0}\right)-\mu}{\sigma} \le r\right] = \lim_{n\to\infty}\left[\frac{\ln\left(\frac{S_n}{S_0}\right)-n\tilde{\mu}}{\tilde{\sigma}\sqrt{n}} \le r\right]$$

$$= \phi(r) \quad (\because \text{central limit theorm})$$

$$\downarrow$$

$$N(0,1)$$

$\Rightarrow \ln\left(\frac{S_n}{S_0}\right) \sim N(\mu, \sigma^2)$ for large n

$\Rightarrow \ln\left(\frac{S_T}{S_0}\right) \sim N(T\mu, T\sigma^2)$ $(\because$ scaling from 1 year to $T)$ for large n

ex: now that $\mathbb{E}[S_T] = S_0 e^{(\mu+\sigma^2/2)T}$ and $\operatorname{var}(S_T) = S_0^2 e^{(2\mu+\sigma^2)T}(e^{\sigma^2 T}-1)$ → double double

Let $W_t := \frac{\ln\left(\frac{S_t}{S_0}\right)-t\mu}{\sigma}$ $t \in [0,T]$

$\Rightarrow W_t \sim N(0,t)$ → wiener process

$$\ln\frac{S_t}{S_0} - t\mu = \sigma W_t$$

$$\Rightarrow \ln\frac{S_t}{S_0} = t\mu + \sigma W_t$$

$$\Rightarrow S_t = S_0 \exp\{t\mu + \sigma W_t\} \to \text{geometric brownian motion (GBT)}$$

$$\Rightarrow dS_t = \tilde{\mu} S_t dt + \sigma S_t dW_t \qquad \tilde{\mu} = \mu + \sigma^2/2$$

ex: show $\tilde{H}_n = \tilde{V}_0 + \sum_{j=0}^{n-1} \theta_j \cdot \Delta \tilde{S}_j$ by induction

Ans: $n=1$:

$$\tilde{V}_0 + \theta_1(\tilde{S}_1 - \tilde{S}_0)$$

$$\tilde{V}_0 = \theta_0 + \phi_0 \tilde{S}_0$$

$$\theta_0 + \phi_0 \tilde{S}_1 = \theta_1 + \phi_1 \tilde{S}_1$$

now $\tilde{V}_1 = \theta_1 + \phi_1 \tilde{S}_1 = \theta_0 + \phi_0 \tilde{S}_1$

$$= \tilde{V}_0 - \phi_0 \tilde{S}_0 + \phi_0 \tilde{S}_1$$

$$= \tilde{V}_0 - \phi_0 \Delta \tilde{S}_0$$

if true for $n+1$ then

$$\tilde{V}_{n+1} = \tilde{V}_0 - \sum_{j=0}^{n} \theta_j \Delta \tilde{S}_j$$

$$\tilde{V}_n = \tilde{V}_{n+1} - \phi_{n+1} \Delta S_{n+1} \quad \text{so we can do the}$$

ex: now that $\mathbb{E}[S_T] = S_0 e^{(\mu + \sigma^2/2)T}$ and $\text{var}(S_T) = S_0^2 e^{(2\mu+\sigma^2)T}(e^{\sigma^2 T} - 1)$

Ans:

$$S_T \sim S_0 e^{N(T\mu, T\sigma^2)}$$

$$\mathbb{E}[e^{tX}] = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\} S_0$$

$$X = N(T\mu, T\sigma^2) \quad t = 1 \Rightarrow \mathbb{E}[S_T] = \exp\left\{T\mu + \frac{\sigma^2 T}{2}\right\} S_0$$

also, $\mathbb{E}[e^{2tX}] - (\mathbb{E}[e^{tX}])^2 = \text{var}(e^{tX})$

$$\Rightarrow \text{var}(S_T) = \left(\exp\{2\mu T + 2\sigma^2 T\} - \left(\exp\left\{\mu T + \frac{\sigma^2 T}{2}\right\}\right)^2\right) S_0^2$$

$$= \left(\exp\{2\mu T + 2\sigma^2 T\} - \exp\{2\mu T + \sigma^2 T\}\right) S_0^2$$

$$= S_0^2 \exp\{2\mu T + \sigma^2 T\}(e^{\sigma^2} - 1)$$

28th march:

Continious time model:

log-normal model to stock price model (GBM)

we want to keep increasing n

$\forall n$, $H_0^{(n)} \to$ option price from replicating portfolio

$\{H_0^{(n)}\}$ conv $\to H_0$

$\Delta t_n = \frac{T}{n}$ $n = 1, 2, \ldots$

the sequence converges and converges to black-scholes formula

$u_n = \exp(\sigma\sqrt{\Delta t_n} + \mu \Delta t_n)$

$d_n = \exp(-\sigma\sqrt{\Delta t_n} + \mu \Delta t_n)$

μ is assumed to be const, r: annualised risk-free interest rate

$$p_n^* = \frac{e^{r\Delta t_n} - d_n}{u_n - d_n}$$

$$\Rightarrow \mathbb{E}^*[H_n^*] = H_0^{(n)}$$

Lemma: $\lim_{n\to\infty} p_n^* = \frac{1}{2}$

proof:

$$p_n^* = \frac{e^{r\Delta t_n} - e^{(-\sigma\sqrt{\Delta t_n} + \mu\Delta t_n)}}{e^{\sigma\sqrt{\Delta t_n} + \mu\Delta t_n} - e^{(-\sigma\sqrt{\Delta t_n} + \mu\Delta t_n)}}$$

$$2p_n^* - 1 = \frac{2e^{r\Delta t_n} - e^{(\sigma\sqrt{\Delta t_n} + \mu\Delta t_n)} - e^{(-\sigma\sqrt{\Delta t_n} + \mu\Delta t_n)}}{e^{(\sigma\sqrt{\Delta t_n} + \mu\Delta t_n)} - e^{(-\sigma\sqrt{\Delta t_n} + \mu\Delta t_n)}}$$

by taylor expansion,

$$2p_n^* - 1 = \frac{\left(r - \frac{\sigma^2}{2} - \mu\right)\Delta t_n + o(\Delta t_n)}{\sigma\sqrt{\Delta t_n}\,(1 + o(1))} \quad \text{as } n \to \infty$$

$o(1) \to 0$ as $\Delta t_n \to 0$

$$\Rightarrow \lim_{n\to\infty} 2p_n^* - 1 = 0$$

$$\Rightarrow p_n^* = \frac{1}{2}$$

Black-Scholes:

$[0, T]$ K

$H_0^{(n)} = \mathbb{E}_{n^*}[e^{-rT} H_T^{(n)}]$

$P_T^{(n)} = \max(0, K - S_T^{(n)})$ $\quad S_T^{(n)} = S_0 e^{X_n}$

X_n is r.v on $(\Omega, \mathbb{P}^*)$

$$X_n(\omega) = \log \prod_{k=1}^{n} \bar{\omega}_k = \sum_{k=1}^{n} Y_k^{(n)}(\omega) \qquad \omega = (\omega_1 \dots \omega_n) \in \Omega$$

$$Y_k^{(n)}(\omega) = \log \bar{\omega}_k \qquad \bar{\omega}_k = \begin{cases} u ; & \omega_k = u \\ d ; & \omega_k = d \end{cases}$$

$$\Rightarrow P_0^{(n)} = \mathbb{E}_n^*\left[e^{-rT} \max(0, K - S_0 e^{X_n}) \right]$$

we know PDE value

Lemma: $\lim_{n\to\infty} \mathbb{E}_n^*(X_n) = \left(r - \frac{\sigma^2}{2}\right)T$

$\lim_{n\to\infty} \mathrm{var}_n^*(X_n) = \sigma^2 T$

proof:

$$\mathbb{P}_n^*\left(\{Y_n^{(n)} = \sigma\sqrt{\Delta t_n} + \mu \Delta t_n\}\right) = p_n^*$$

$$\mathbb{P}_n^*\left(\{Y_n^{(n)} = -\sigma\sqrt{\Delta t_n} + \mu \Delta t_n\}\right) = 1 - p_n^*$$

$$\mathbb{E}_n^*[Y_k^{(n)}] = p_n^*(\sigma\sqrt{\Delta t_n} + \mu \Delta t_n) + (1 - p_n^*)(-\sigma\sqrt{\Delta t_n} + \mu \Delta t_n)$$

$$= (2p_n^* - 1)\sigma\sqrt{\Delta t_n} + \Delta t_n(\mu p_n^* + \mu(1 - p_n^*))$$

$$= \frac{\left(r - \frac{\sigma^2}{2} - \mu\right)\Delta t_n + o(\Delta t_n)}{1 + o(1)} + \cancel{\Delta t_n \mu}$$ by using $2p_n^* - 1$ as previous

as $n \to \infty$:

$$X_n = \sum_{k=1}^{n} Y_k^{(n)}$$

$$\mathbb{E}^*[X_n] = n\, \mathbb{E}_n^*[Y_1^{(n)}]$$

$$= n\left\{\left(r - \frac{\sigma^2}{2}\right)\Delta t_n + o(\Delta t_n)\right\} \qquad \text{as } n \to$$

($\Delta t_n \to \frac{T}{n}$)

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \left(r - \frac{\sigma^2}{2}\right)T$$

End: Slow above lemma for variance → does down

Lemma: the seq of r.v $\{X_n\}$ conv in distribution to

$$X \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

Theorem: (BSM for put) Let $P_0^{(n)}$ price of European-strike $[0,T]$

$$\lim_{n\to\infty} P_0^{(n)} = P_0$$

$$P_0 = \mathbb{E}[e^{-rT} P_T]$$

$$P_T = \max(0, K - S_0 e^X) \qquad X \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

Def: (BS Price for put) The number P_0 defined by above theorm

$$P_0 = \mathbb{E}[e^{-rT} P_T]$$

is called BSP for European put option with strike K and maturity T

Lem: (BS Formula for put) $P_0 = Ke^{-rT}\phi(-d_-) - S_0\phi(-d_+)$

ϕ is normal dist

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2}\, ds \quad x \in \mathbb{R}$$

$$d_\pm = \frac{\log\left(\frac{S_0}{K}\right) \pm \left(r \pm \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

r is prevaig intest rate

from put call parity, in option crain we can get r

all ther parameter roun

then we use Newton-Raphson to get σ

Exe: Show above lemma for variance

Ans:

$$\mathbb{E}[X_n] = \left(r - \frac{\sigma^2}{2}\right)T$$

$$\mathbb{E}[X_n^2] - \left(\mathbb{E}[X_n]\right)^2$$

$$= \mathbb{E}\left[\sum (Y_n)^2 + 2\sum\sum Y_n Y_m\right] - \left(\mathbb{E}[X_n]\right)^2$$

$$= n\,\mathbb{E}[Y_n^2] + 2\underbrace{\binom{n}{2}}_{(n)(n-1)=n^2-n}\left(\mathbb{E}[Y_n]\right)^2 - n^2\mathbb{E}[Y_n]^2$$

$$= n\,\mathbb{E}[Y_n^2] - n\,\mathbb{E}[Y_n]^2$$

$$= n\left[p_n^*\left(\sigma\sqrt{\Delta t_n} + \mu\Delta t_n\right)^2 + (1-p^*)\left(-\sigma\sqrt{\Delta t_n} + \mu\Delta t_n\right)^2\right]$$

$$- n\left(\frac{\left[\left(r - \frac{\sigma^2}{2} - \mu\right)\Delta t_n + o(\Delta t_n) + \Delta t_n \mu\right]}{1 + o(1)}\right)^2$$

$$= n\left[p_n^*\left(\sigma^2\Delta t_n + \mu^2\Delta t_n^2 + 2\sigma\sqrt{\Delta t_n}\,\mu\Delta t_n + \sigma^2\Delta t_n + \mu^2\Delta t_n^2 - 2\sigma\sqrt{\Delta t_n}\,\Delta t_n\right)\right.$$

$$\left. + o(\Delta t_n)\right]$$

$$= p_n^*(2\sigma^2 T) + o(\Delta t_n)$$

as $n \to \infty$ $p_n^* \to \frac{1}{2}$ $o(\Delta t_n) \to 0$

$\Rightarrow \operatorname{var}(X_n) = \sigma^2 T$

1st April:

Theorem: (BSM for put) Let $P_0^{(n)}$ be price of european k-strike put option on $[0,T]$

$$\lim_{n\to\infty} P_0^{(n)} = P_0$$

$$P_0 = \mathbb{E}^*[e^{-rT} P_T]$$

$\mathbb{E}^*$ is expectation in risk-neutral probability measure

$$P_T = \max(0, K - S_0 e^X)$$

$$X \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right)$$

Lem: (Black scholes formula for european call) $P_0 = Ke^{-rT}\phi(-d_-) - S_0\phi(-d_+)$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-x^2/2}\,ds \quad x \in \mathbb{R}$$

$$d_\pm = \frac{\log\left(\frac{S_0}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Note: from put option price, and put-call parity we get call option price

$\frac{S_t}{K}$ = moneyness

$$P_0 = \mathbb{E}^*[e^{-rT} P_T] \Leftrightarrow P_0 = Ke^{-rT}\Phi(-d_-) - S_0\Phi(-d_+)$$

$$d_\pm = \frac{\log\left(\frac{S_0}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

as, $P_T = \max(0, K - S_T)$

$$\Rightarrow P_0 = e^{-rT}\mathbb{E}^*[\max(0, K - S_T)]$$

$$= e^{-rT}\mathbb{E}^*[K\,\mathbb{I}_{S_T<K}] - e^{-rT}\mathbb{E}^*[S_T\,\mathbb{I}_{S_t<K}]$$

now, $S_T < K \Leftrightarrow Z < -d_-$; $Z \sim N(0,1)$

$S_T = S_0 e^X$ $\quad X \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right)$ $\quad -d_- = -\frac{\log\left(\frac{S_0}{K}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

now, $S_T < K \Leftrightarrow S_0 e^X < K$

$$\Leftrightarrow S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right) < K$$

$$\Leftrightarrow \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z < \log\frac{K}{S_0}$$

$$\Leftrightarrow Z < \frac{-\log\frac{S_0}{K} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}Z} = -d_-$$

now, $\mathbb{E}^*\left(\mathbb{I}_{S_T<K}\right) = \mathbb{E}^*\left(\mathbb{I}_{Z<-d_-}\right) = \Phi(-d_-)$

now, $E^*[S_T \mathbb{I}_{S_T<K}] = E^*[S_0 e^X \mathbb{I}_{Z<-d_-}]$

$= S_0 E^*[\exp((r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z)\mathbb{I}_{Z<-d_-}]$

$= S_0 e^{rT} E^*[\exp(-\frac{\sigma^2}{2}T + \sigma\sqrt{T}Z)\mathbb{I}_{Z<-d_-}]$

$= S_0 e^{rT} \int_{-\infty}^{-d_-} \exp\left(-\frac{\sigma^2}{2}T + \sigma\sqrt{T}s\right)\exp\left(\frac{-s^2}{2}\right) ds$

usage of variable $r = s - \sigma\sqrt{T}$

$\Rightarrow -\frac{\sigma^2}{2}T + \sigma\sqrt{T}s - \frac{s^2}{2} = -\frac{\sigma^2}{2}T + \sigma\sqrt{T}(r+\sigma\sqrt{T}) - \frac{(r+\sigma\sqrt{T})^2}{2}$

$= -\frac{\sigma^2}{2}T + \sigma r\sqrt{T} + \sigma^2 T - \frac{(r^2 + \sigma^2 T + 2r\sigma\sqrt{T})}{2}$

$= -\frac{r^2}{2}$

$= S_0 e^{rT} \int_{-\infty}^{-d_- - \sigma\sqrt{T}} \exp\left(-\frac{r^2}{2}\right) dr$

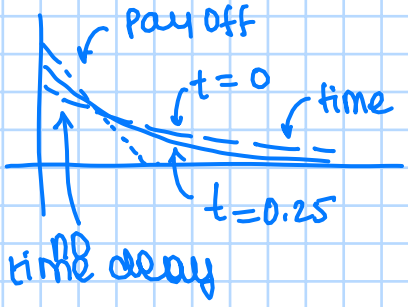
$= S_0 e^{rT} \Phi(-d_- - \sigma\sqrt{T})$

$-d_+ = -\frac{\ln\left(\frac{S_0}{K}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

$= -\frac{\ln\left(\frac{S_0}{K}\right) - \left(r - \frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}}{\sigma\sqrt{T}}$

$= -d_- - \sigma\sqrt{T}$

so, $P_0 = Ke^{-rT}\Phi(-d_-) - S_0\Phi(-d_+)$



conv: (Black-scholes for call) $C_0 = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-)$ where $S_0, K, r, \Phi(x), d\pm$ are as above

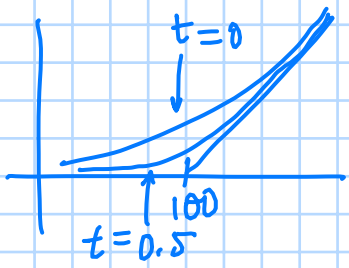
proof: $C_0 - P_0 = S_0 - Ke^{-rT}$

$C_0 = -S_0\Phi(-d_+) + Ke^{-rT}\Phi(-d_-) + S_0 - Ke^{-rT}$

$= S_0[1 - \Phi(-d_+)] - Ke^{-rT}[1 - \Phi(-d_-)]$

$= S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-)$

as, $1 = \Phi(x) + \Phi(-x)$



now, Vol = 100/65/50 units
K will be given, underlying, volatility in % $\sigma = \frac{Vol}{100}$

we can also derive a parabolic partial differential equation as a limiting case of binomial model

we assumed completness (we have a replicating portfolio)

$H_0 = \mathbb{E}^*[H_T^*]$
↑ R-neutral ← discounted

Self-financing: $V_0 = \phi_1 B_0 + \theta_1 S_0 = \mathbb{E}^*(e^{-rT} H_T)$
↑
replicating portfolio

as
$V_T \sim H_T$ ← at T it meets our obligation

so, V_0 will be our price of the option

moreover, if we delta hedge $\theta_1 = \frac{H^u - H^d}{S_0(u-d)} \approx \frac{\partial H}{\partial S} = \Delta$, then by replicating portfolio we can get same C

we will get $\frac{\partial H}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 H}{\partial S^2} + rS\frac{\partial H}{\partial S} - rH = 0$

$\{\pi_t\}$, $\pi_t = (\phi_t, \theta_t, h_t)$ ← strategy (any measurable process)

$V_t = \phi_t B_t + \langle \theta_t, S_t \rangle + \langle h_t, H_t \rangle$

def: (self-financing) $dV_t = \phi_t B_t + \theta_t \cdot dS_t + h_t \cdot dH_t$

Delta-Hedging:

let $(\phi_t, \theta_t)_{0 \le t \le T}$ be self-financing strategy

$H = H(S_T)$ payoff of european contingent claim at T

$V_T = H(S_T) \Rightarrow V_t = H_t \quad t \le T$ (∵ law of one price)
$V_T = V(t, S_t)$ for some sufficiently smooth function $V(t,S)$

$dV_t = \phi_t dB_t + \theta_t dS_t$ (∵ self-financing)

$dB_t = rB_t dt$ (∵ risk-neutral)

$dS_t = \mu S_t dt + \sigma S_t dW_t \Rightarrow dV_t = (\phi_t r B_t + \theta_t \mu S_t)dt + \sigma S_t dW_t$

by Ito's lemma: $dH_t = \left(\frac{\partial H_t}{\partial t} + \mu S_t \frac{\partial H_t}{\partial S} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 H_t}{\partial S^2}\right) dt + \left(\sigma S_t \frac{\partial H_t}{\partial S}\right) dW_t$

as $H_t = H(S_t, t)$

now $dV_t = dH_t$

$\Rightarrow \phi_t r B_t + \theta_t \mu S_t = \frac{\partial H_t}{\partial t} + \mu S_t \frac{\partial H_t}{\partial S} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 H_t}{\partial S^2}$

and $\sigma S_t \frac{\partial H_t}{\partial S} = \sigma S_t \theta_t$

$\Rightarrow \frac{\partial H_t}{\partial S} = \theta_t$ and $\phi_t r B_t + \theta_t \mu S_t = \frac{\partial H_t}{\partial t} + \theta_t \mu S_t + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 H_t}{\partial S^2}$

$$\phi_t = \frac{\frac{\partial H_t}{\partial t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 H_t}{\partial S^2}}{r B_t}$$

now $V_t = \phi_t B_t + \theta_t S_t$

$\Rightarrow V_t = \left(\frac{\partial H_t}{\partial t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 H_t}{\partial S^2}\right) \frac{B_t}{r B_t} + \left(\frac{\partial H_t}{\partial S}\right) S_t$

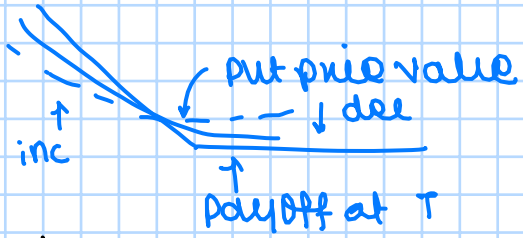
$\Rightarrow rV = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S}$ as $H = V$

so we get $\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$

8th April:

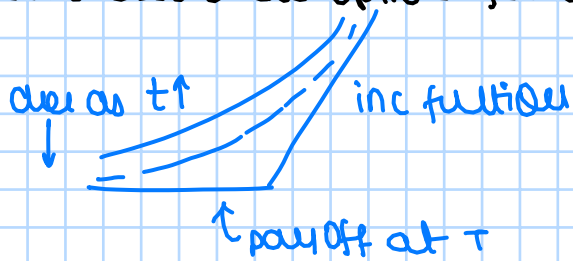
Defn: (Black-scholes price for put) $P_0 = \mathbb{E}^*[e^{-rT} P_T]$

we have also derived $P_0 = Ke^{-rT}\phi(-d_-) - S_0\phi(-d_+)$



Note: $\forall t < T$, the put option price is a smooth function

we also derive call option formula using put call parity



limiting case of binomial model

$$\frac{\partial H}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 H}{\partial S^2} + rS\frac{\partial H}{\partial S} - rH = 0$$

the starting level of backward induction at time $t = t_n = T$ can be

$$H = H(t, S)$$
$$H(T, S) = H_T \text{ payoff of european option}$$

another way to derive by delta hedging

$$\{\pi_t\},\ \pi_t = (\phi_t, \vec{\theta}_t, \vec{h}_t)$$
$$V_t = \phi_t B_t + \vec{\theta}_t \cdot \vec{S}_t + \vec{h}_t \cdot \vec{H}_t$$

Defn: A strategy is said to be self-financing if

$$dV_t = \phi_t dB_t + \vec{\theta}_t \cdot d\vec{S} + \vec{h}_t \cdot d\vec{H}_t$$

as, $V_k(\pi_k) = V_k(\pi_{k+1}) \Rightarrow \phi_k B_k + \theta_k S_k = \phi_{k+1} B_k + \theta_{k+1} B_k$

$$\Rightarrow \Delta\phi_k B_k + \Delta\theta_k S_k = 0$$

$$dV_t = \phi_t dB_t + \theta_t dS_t + d\phi_t B_t + d\theta_t S_t + d\phi_t dB_t + d\theta_t dS_t$$

Ito we get extra term

Becomes 0 as when there is small change in price, it will lead to small change in θ so, very small term

$$\Rightarrow dV_t = \phi_t dB_t + \theta_t dS_t$$

Delta-Hedging :

let $(\phi_t, \theta_t)_{0 \le t \le T}$ be self financing trading strategy

$H = H(S_T)$ be payoff of european claim at maturity T

$$V_t = \phi_t B_t + \theta_t S_t$$

$$dV_t = \phi_t dB_t + \theta_t dS_t$$

$$dB_t = rB_t dt \qquad dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$\Rightarrow dV_t = (\mu S_t dt + \sigma S_t dW_t)\theta_t + (rB_t)dt\,\phi_t \quad \text{—— ①}$$

also by Ito's lemma on $H = H(t, S)$

$$dH_t = \left(\frac{\partial H}{\partial t} + \mu S_t \frac{\partial H}{\partial S} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 H}{\partial S^2}\right) dt + \sigma S_t \frac{\partial H}{\partial S} dW_t \quad \text{— ②}$$

as $dV_t = dH_t$, equating ①, ②

$$\theta_t \cancel{S_t} = \cancel{S_t} \frac{\partial H}{\partial S}$$

delta: $\theta_t = \frac{\partial H}{\partial S}$ —— ③

and $\frac{\partial H}{\partial t} + \mu S_t \cancel{\frac{\partial H}{\partial S}} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 H}{\partial S^2} = \mu S_t \theta_t + r B_t \phi_t$

$$\phi_t = \left(\frac{\partial H}{\partial t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 H}{\partial S^2}\right) \frac{1}{rB_t} \quad \text{—— ④}$$

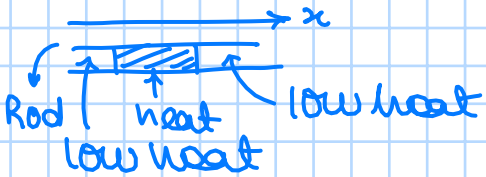
now, as $V_t = \phi_t B_t + \theta_t S_t$

but $V_t = H_t = \phi_t B_t + \theta_t S_t$

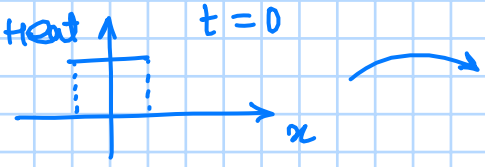
$$\phi_t = \frac{H_t - \theta_t S_t}{B_t} \quad \text{— ⑤}$$

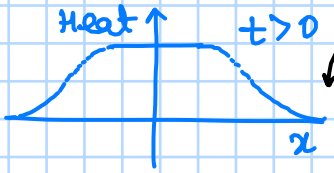
then equating ④, ⑤

$$\frac{\partial H}{\partial t} + rS \frac{\partial H}{\partial S} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 H}{\partial S^2} = rH_t$$



$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{(Heat equation)}$$





smooth function

so, smoothing effect is coming from $\frac{\sigma^2}{2} S_t^2 \frac{\partial^2 H}{\partial S^2}$ term

$H = C(\text{Call})$, $H = P(\text{Put})$, terminal $H(T,S) = H_T(S)$ payoff

if we solve the solution, then that is price

$$H(t,S) = Ke^{-r(T-t)}\phi(-d_-) - S_t\phi(-d_+)$$

$$d_- = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_+ = d_- + \sigma\sqrt{T-t}$$

Exe: show that as $t \to T$, the $H(t,S)$ becomes put payoff → done down

Initial value problem:

$H(t,S) = u(\tau, x)$

$\tau = T - t \quad x = \log S$

Martingale pricing:

Defn: A stocastic process $\{M_t\}$ adapted to $\{\mathcal{F}_t\}$ is called a martingale if

$$\mathbb{E}[M_t | \mathcal{F}_s] = M_s \quad \forall s \le t$$

if a filteration is generated from brownian motion, then it is called a Brownian filteration

we showed that if S_t was a binomial tree structe, then $\{\tilde{S}_t\}$ is martingale

we limit the binomial model and on a brownian filteration we loose martingale property of $\{\tilde{S}_t\}$ (not Brownian martingale)

$\int f(s)dw$

↳ wiener integral

↳ ito if $f(s)$ is a RV

Note: All properties and ideas are same as wiener integral

Lemma: let $\{W_t^{\mathbb{P}}\}$ brownian motion on space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{b_t\}$ be an adapted process satisfying

$$\mathbb{E}\left[\int_0^T b_s^2 ds\right] < \infty$$

then the process $M_t := M_0 + \int_0^t b_s dW_s$ is a brownian martingale

$dM_t = b_t dW + 0 dt$

↑ vol ↑ drift

Theorem: (Martingale Representation Theorem) Let $\{M_t\}$ be a square integrable

brownian martingale wrt brownian motion $\{W_t^{\mathbb{P}}\}$ then $\exists$ unique square integrable process $\{b_t\}$ adapted to Brownian filtration s.t

$$M_t = M_0 + \int_0^t b_s dW_t \quad a.s \ \forall \ t \in [0,T]$$

so, an ito process of form $dX_t = b_t dW_t$

$\Leftrightarrow$ X_t is martingale

but, $\{S_t\}$ $dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}$

$W_t^{\mathbb{P}}$: Brownian motion under physical measure $\mathbb{P}$

so, $\tilde{S}_t = e^{-rt} S_t$

$$d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t^{\mathbb{P}}$$

if $\mu = r$ then martingale

Exe: Show that as $t \to T$, the $H(t,S)$ becomes put payoff

Ans:

$$H(t,S) = Ke^{-r(T-t)}\phi(-d_-) - S_t\phi(-d_+)$$

$$d_- = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_+ = d_- + \sigma\sqrt{T-t}$$

now, as $t \to T$ $\lim_{t\to T} d_+ = \lim_{t\to T} d_- = \begin{cases} -\infty : S_T > K \\ +\infty ; S_T \leq K \end{cases}$

$$\lim_{t\to T} H(t,S) = \begin{cases} K\phi(-\infty) - S_t\phi(-\infty) ; S_T > K \\ K\phi(+\infty) - S_t\phi(+\infty) ; S_T \leq K \end{cases}$$

$$= \begin{cases} 0 ; S_T > K \\ K - S_T ; S_T \leq K \end{cases}$$

$$= \max(0, K - S_T)$$

10th April:

$$\frac{\partial H}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 H}{\partial S^2} + rS\frac{\partial H}{\partial S} - rH = 0 \qquad H(T,S) = H_T(S)$$

for put, $H(t,S) = Ke^{-r(T-t)}\phi(-d_-) - S_t\phi(-d_+)$

$$d_- = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \qquad d_+ = d_- + \sigma\sqrt{T-t}$$

now, we want to know if $H(T,S) = H_T(S)$ is the only solution or not (ie unique solution)

Defn: A process $\{M_t\}$ adapted to $\{\mathcal{F}_t\}$ is called martingale if

$$\mathbb{E}[M_t | \mathcal{F}_s] = M_s \quad \forall\, s \le t$$

filteration generated by brownian motion, then it is called brownian filtation

Lemma: $\{W_t^{\mathbb{P}}\}$ brownian motion $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{b_t\}$ be an adapted process with

$$\mathbb{E}\left[\int_0^T b_s^2\, ds\right] < \infty$$

then $M_t := M_0 + \int_0^t b_s\, dW_s$ is a brownian martingale

Theorem: (Martingale Representation) Let $\{M_t\}$ be square integrable brownian martingale wrt brownian filteration $\{W_t^{\mathbb{P}}\}$, then $\exists$ a unique square-integrable process $\{b_t\}$ adapted to $\{W_t^{\mathbb{P}}\}$ s.t

$$M_t = M_0 + \int_0^t b_s\, dW_t^{\mathbb{P}} \quad a.s \quad \forall\, t \in [0,T]$$

$dM_t = b_t\, dW_t^{\mathbb{P}}$ $\{X_t\}$ ito process $\Rightarrow dX_t = \mu(X,t)dt + \sigma(X,t)dW_t^{\mathbb{P}}$

M_t is ito process s.t $\mu(X,t) \equiv 0$ i.e

$$dM_t = 0.dt + b_t dW_t^{\mathbb{P}}$$

but $dS_t = \mu S_t dt + \sigma S_t dt$

$\neq 0 \Rightarrow S_t$ is not a martingale

but $\tilde{S}_t = e^{-rt} S_t$

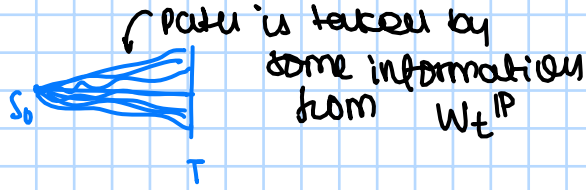
$\Rightarrow d\tilde{S}_t = (\mu - r)\tilde{S}_t\, dt + \sigma \tilde{S}_t dW_t$

so, in discounted setup, there is a scope to make $\tilde{S}_t$ a martingale

$\mu = r$

$\Rightarrow d\tilde{S}_t = \sigma \tilde{S}_t dW_t$

$\Rightarrow dS_t = rS_t dt + \sigma S_t dW_t$



Eg: $\mu = 0.05$ $\sigma = 0.2$ then $\mathbb{P}(\{S_t > 1.25\})$ μ, σ plays a role if we change μ

we wage $\mathbb{P}(\{S_t > 1.25\})$, but for option it does not matter as we are trying to hedge risk

Note: In real world, $\tilde{S}_t$ not a martingale

Theorem: (Girsanov) Let $\{W_t^{\mathbb{P}}\}$ be a Brownian motion on $(\Omega, \mathcal{H}, \mathbb{P})$ w.r.t $\{\mathcal{H}_t\}$ Let $\{\theta_t\}$ be an adapted process

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)\right] < \infty$$

$\{Z_t\}$ by $Z_t := \exp\{-\int_0^t \theta_s dW_s^{\mathbb{P}} - \frac{1}{2}\int_0^t \theta_s^2 ds\}$

$\{Z_t\}$ martingale under $\mathbb{P}$, Let $\mathbb{Q}$ on $\mathcal{H}_T$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$$

then the process $\{W_t^{\mathbb{Q}}\}$ $W_t^{\mathbb{Q}} := W_t^{\mathbb{P}} + \int_0^t \theta_s ds$ is a Brownian motion on $(\Omega, \mathcal{H}, \mathbb{Q})$ wrt filteration $\{\mathcal{H}_t\}$

So, intuitively $dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}$

$\Rightarrow dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$ from above theorem

$W_t^{\mathbb{P}} - W_s^{\mathbb{P}} \sim N(0, t-s)$ under $\mathbb{P}$

$W_t^{\mathbb{Q}} - W_s^{\mathbb{Q}} \sim N(0, t-s)$ under $\mathbb{Q}$

$\theta_s = \frac{\mu - r}{\sigma}$ then $\exists$ $\mathbb{Q}$ prob measure s.t

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t \theta_s ds$$

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \left(\frac{\mu - r}{\sigma}\right) t$$

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \left(\frac{\mu - r}{\sigma}\right) dt$$

So, $d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t^{\mathbb{P}}$

$$= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t \left(dW_t^{\mathbb{Q}} - \left(\frac{\mu - r}{\sigma}\right) dt\right)$$

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t^{\mathbb{Q}}$$

$$\Rightarrow dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

So, wrt $\mathbb{Q}$, $\{\tilde{S}_t\}$ is a martingale

Feynman-Kac formula:

Let $H(S_T)$ denote payoff of a european derivate at time T, under risk neutral-measure $\mathbb{Q}$,

$$V_t := \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} H(S_T) \mid \mathcal{H}_t^{\mathbb{Q}}]$$

then the feynman-kac gives that V_t is unique

Theorm: (Feynman-Kac) Let $\{W_t^{\mathbb{Q}}\}$ be a brownian motion under the risk neutral measure $\mathbb{Q}$ and let $\{S_t\}_{t\geq 0}$ satisfy

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}} \quad S_t = s > 0$$

Let $H_T : [0,\infty) \to \mathbb{R}$ be a continious payoff futiion satisfying a polynomial growth condition, i.e

$$|H_T(S)| \leq C(1+S^m) \quad \text{for some } C, m > 0$$

Let $V_t = V(t,S)$ be above, then the following holds:

existance: The futiion $V(t,S)$ is a classical solution $V \in C^{1,2}([0,T) \times (0,\infty))$ of black scoles PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad 0 < t$$

with tominal condition $V(T,S) = H_T(S)$

Uniqueness: The futiion $V(t,s)$ is the unique solution in the class of futiions with almost polynomical growth in s

15th April:

we have already seen our formula as fution of variables

partial derivatives of option prices are called greeks

let V denote the price of an option or the value of a portfolio

$\Pi = (0, 0, h)$ ← $h = (c_1, c_2, p_1, p_2, \dots)$ (options) → all options are underlying

$V_\Pi = \langle h, H \rangle$ ← we can also talk about greeks of V as it satisfies boundaries equation (∵ it is a linear PDE) (or save underlying)

$\frac{\partial V}{\partial S} = \Delta \Rightarrow$ delta V → value of the portfolio in particular, it can be

sensitivity of V wrt stock price a call/put

$\frac{\partial^2 V}{\partial S^2} = \Gamma \Rightarrow$ gamma, sensitivity of Δ wrt S

$\frac{\partial V}{\partial t} = \Theta \Rightarrow$ theta

$\frac{\partial V}{\partial \sigma} = \nu \Rightarrow$ vega

↑ implied volitility

$\frac{\partial V}{\partial r} = \rho \Rightarrow$ rho

Delta:

$\Delta_V := \frac{\partial V}{\partial S}$, we have the formula $c(t, S, r, \sigma) = S\phi(d_+) - Ke^{-r(T-t)}\phi(d_-)$

$P(t, S, r, \sigma) = Ke^{-r(T-t)}\phi(-d_-) - S\phi(-d_+)$

$$d_\pm = \frac{\log\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Note: $g'(d_\pm) = g(d_\pm)$

$$\Delta_c := \frac{\partial c}{\partial S} = \phi(d_+)$$

$$\Delta_p := \frac{\partial p}{\partial S} = -\phi(-d_+)$$

given Δ_c, we can use put call parity to get Δ_p

$$p - c = Ke^{-r(T-t)} - S_t$$

$$\Delta_p - \Delta_c = 0 - \frac{\partial S_t}{\partial S} = -1$$

$$\Delta_p = -1 + \Delta_c$$

this when we want self financing

In a delta Neutral portfolio, $\Pi = (0, 0, h)$

$$V = \theta S + hH$$
$$\frac{\partial V}{\partial S} = \theta + h\frac{\partial H}{\partial S}$$
$$= \theta + h\Delta_S$$

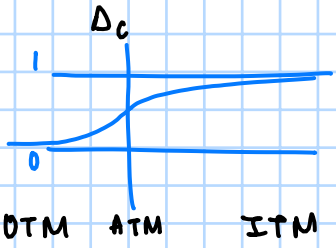
Given Δ_H, given h, find θ

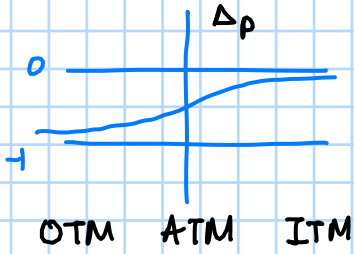
$$0 = \theta + h\Delta_S$$
$$\theta = -h\Delta_S$$

Or given θ, find h:

$$h = -\frac{\theta}{\Delta_S}$$

↑ this is delta hedging





Def: (Delta neutral) A portfolio is called delta neutral at some time $t \in [0,T]$ if $\Delta_V = 0$ at t

Eg: $\Delta_C = 0.6$ lot size = 100, trader sold 20 call option contracts

$$\Pi = (0, S, -100 \times 20)$$
$$\Pi = (0, S, -2000)$$

then $S + \Delta_C(-2000) = 0$ $\quad V = S - 2000 \times C$

$$S = 2000 \times 0.6$$
$$= 1200$$

now, most calls ⇒ positive delta
Hedge by taking long position in shares

portfolio is delta-Neutral at that time instant

Eg: $\Delta_C = 0.7$ new delta

$S = 1200$ existing

$$\Pi_2 = (0, S, -2000)$$

then $S = 2000 \times \Delta_C$
$= 1400$

now $S = 1200$ ⇒ 200 more shares/futures buy

with existing hedge, $\Delta_V = 1200 - 2000 \times \Delta_C = -200$ ← overall short

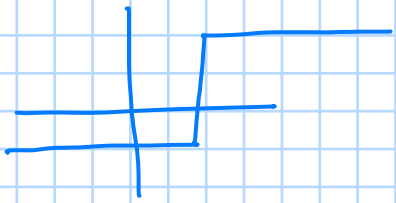
so, we can do 200 more and be done

Dynamic Hedging:

Hedge must be adjusted as delta

Eg: $K_1 = 22{,}150$ $C_{1,0} = 89$ } same underlying
$K_2 = 22{,}250$ $C_{2,0} = 48$ }

$\Pi_1 = (0, \Theta_1, (C_1 - C_2))$ at $t = t_0$



$C_1 = C_2 = C$ (bull spread) $t = t_0$

$\Pi_1 = (0, \Theta_1, (C_1 - C))$

↑ choose Θ s.t we become delta neutral

$t_0 < t_1 < T$ K_1 $C_{1,1} = 155$ $\Delta_{1,1} = 0.53$
K_2 $C_{2,1} = 65$ $\Delta_{2,1} = 0.44$

$$\Theta + C\Delta_{1,1} - C\Delta_{2,1} = 0$$

$$\Theta = C[\Delta_{2,1} - \Delta_{1,1}] = -0.09 \times C$$

but we can also do option adjustment

$\Pi_1 = (0, \Theta_1, (1, -1))$ at $t = t_0$

↳ $S = 0$, we now find C_2 i.e $V_2 = cC_1 - C_2C_2$

↑ we wage this to become delta Neutral



$$0 = c\Delta_{1,1} - c_2\Delta_{2,1}$$

$$C_2 = C\frac{\Delta_{1,1}}{\Delta_{2,1}} = C\left(\frac{53}{44}\right) \approx 1.2C$$

Gamma:

$\frac{\partial^2 V}{\partial S^2}$, people don't trade when gamma is large

more Γ, we have to adjust portfolios very often

$V = (0, \Theta, (C_1, C_2))$

$\frac{\partial V}{\partial S} = \Theta + C_1\Delta C_1 + C_2\Delta C_2$ $\frac{\partial^2 V}{\partial S^2} = C_1\Gamma_{C_1} + C_2\Gamma_{C_2}$

$\Theta + C_1\Delta C_1 + C_2\Delta C_2 = 0$
$C_1\Gamma_{C_1} + C_2\Gamma_{C_2} = 0$ } delta-gamma Hedging

$\Gamma_C = \frac{e^{-d^2/2}}{\sqrt{2\pi(T-t)}S\sigma}$, by put call parity

$\Gamma_P - \Gamma_C = 0 \Rightarrow \Gamma_P = \Gamma_C$

17th April:

we have seen greeks as derivative of C, P option price

$\frac{\partial V}{\partial S} := \Delta$ Delta

$\frac{\partial^2 V}{\partial S^2} := \Gamma$ Gamma

$\frac{\partial V}{\partial t} := \Theta$ Theta

$\frac{\partial V}{\partial \sigma} := \nu$ Vega

$\frac{\partial V}{\partial r} := \rho$ Rho

we can hedge with Δ, Γ, ν usually Θ we cannot directly hedge, and we don't mind ρ

we have also seen how to hedge with a greek, i.e position in option has to be adjusted with position in underlying

only with options (portfolio adjustment) to get greek neutral portfolio also possible

$\Delta_C > 0, \quad \Delta_P < 0$

$\Gamma_C > 0, \Gamma_P > 0$

$\Pi = (\underbrace{O}_{\text{Bank account}}, \underbrace{S}_{\text{underlying}}, \overbrace{h}^{\text{option}})$

$V = \Theta S + h_1 H_1 + h_2 H_2$

$\Delta_V = \Theta + h_1 \Delta_{H_1} + h_2 \Delta_{H_2}$

$\Delta_V = 0$

also $\Gamma_V = h_1 \Gamma_{H_1} + h_2 \Gamma_{H_2}$

$\Gamma_V = 0$ for 2 variables, it is better to have gamma as small as possible

Delta-gamma Hedge:

K_1 C_1 $\quad$ K_2 C_2 } on same underlying

$\Delta_V = \Theta + h_1 \Delta_{H_1} + h_2 \Delta_{H_2}$

$\Gamma_V = h_1 \Gamma_{H_1} + h_2 \Gamma_{H_2}$

$\Delta V = 0, \ \Gamma_V = 0 \Rightarrow \begin{pmatrix} \Delta_{H_1} & \Delta_{H_2} \\ \Gamma_{H_1} & \Gamma_{H_2} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} -\Theta \\ 0 \end{pmatrix}$

$\Gamma_C = \frac{e^{\frac{-d_+^2}{2}}}{\sqrt{2\pi(T-t)}\,S\sigma} = \Gamma_P$

as $S - C = Ke^{-rT} - S_t$

$\Delta_S - \Delta_C = 1$

$\Gamma_S - \Gamma_C = 0 \Rightarrow \Gamma_C = \Gamma_S$

Theta:

$\theta = \frac{\partial V}{\partial t}$ for option buyers, avoid to have a very negative θ

$\theta_c, \theta_p, \theta_v$ for call, put, portfolio

$$\theta_c = -\frac{S\sigma}{2\sqrt{2\pi(T-t)}} e^{-d_+^2/2} - rKe^{-r(T-t)}\Phi(d_-)$$

we can use black-scholes to derive this

$$\frac{\partial H}{\partial t} + rS\frac{\partial H}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 H}{\partial S^2} - rH = 0$$

$$\Rightarrow \theta_c + rS\Delta_c + \frac{\sigma^2}{2}S^2\Gamma_c - rC = 0$$

$$\Rightarrow \theta_c = rC - rS\Delta_c - \frac{\sigma^2}{2}S^2\Gamma_c$$

so, we can derive it or use black-scholes to derive it

Vega:

$\nu = \frac{\partial V}{\partial \sigma}$ $\quad \nu_c = \frac{S\sqrt{T-t}}{\sqrt{2\pi}} e^{-d_+^2/2}$

but $\nu_c = \nu_p$ as

$$P - C = Ke^{-r(T-t)} - S$$

$$\Rightarrow \nu_p - \nu_c = 0$$

$$\Rightarrow \nu_c = \nu_p$$

for a portfolio $\nu_v = \theta\frac{\partial S}{\partial \sigma} + c\nu_c + p\nu_p = 0$

↑ vega neutral portfolio

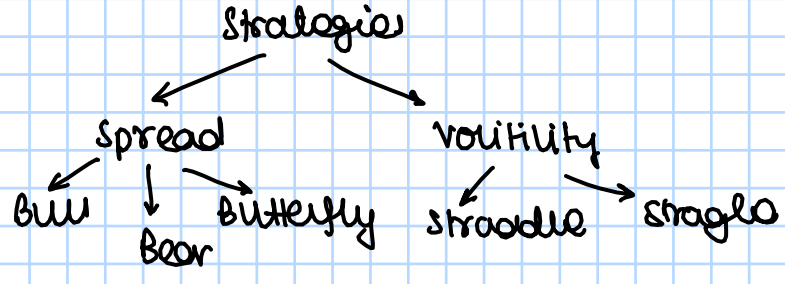
$$c = -p\frac{\nu_p}{\nu_c} \text{ or } p = -c\frac{\nu_c}{\nu_p}$$

Rho:

$\rho = \frac{\partial V}{\partial r}$

Trading strategies:

$\pi = (0, 0, \vec{n})$

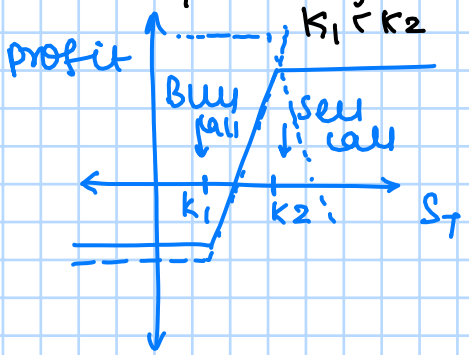


when we have positive sentiment → Bull spread
similarly Bear / Butterfly

vertical spread → same expiry
Horizontal spread / calendar spread → Horizontal spread

Bull spread:

Buying call option at K_1 - strike, selling K_2 strike call



$V(S_T)$ = Net gain function

$$= \max(0, S_T - K_1) - X_1 + (X_2 - \max(0, S_T - K_2))$$

$$= \begin{cases} X_2 - X_1 & ; S_T \le K_1 \\ S_T - K_1 + X_2 - X_1 & ; K_1 \le S_T \le K_2 \\ K_2 - K_1 + X_2 - X_1 & ; K_2 \le S_T \end{cases}$$

Similarly Bear spread

Butterfly:

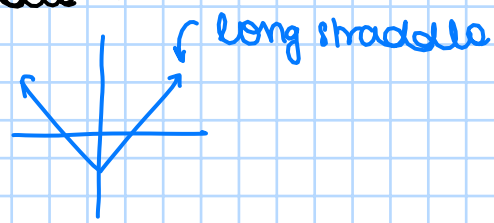
$0 < K_1 < K_2 < K_3$ $\quad K_2 \in [S_0 - \varepsilon, S_0 + \varepsilon]$ for some sufficiently small $\varepsilon > 0$

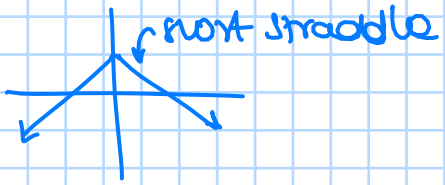
buy K_1, K_2 call and sell $2K_2$



Volitility speculation:

straddle:





Strangle:

